NUMERICAL METHODS USED TO MODEL UNSTEADY CANAL FLOW

By T. S. Strelkoff and H. T. Falvey, Members, ASCE

ABSTRACT: This paper presents a critical review of numerical methods used to model unsteady flow in canals. The significance of the various forms of the governing equations is outlined, and the problems associated with the evaluation of boundary drag and head loss are introduced. The attributes of the numerical solution techniques are described. These attributes include applicability, accuracy, convenience, and robustness. Both legitimate and nonlegitimate methods of achieving robustness are considered. Approximate hydrologic techniques are viewed from the perspective gained by a review of the complete equations. Characteristic and finite-difference techniques for solving the full equations are compared. Practical difficulties in detecting bore-wave formation during a simulation are noted. Specific techniques are recommended for difficult problems such as the computation of very shallow flows. Practical considerations concerning testing of the techniques and a cautionary note for users of computer-simulation models are given.

INTRODUCTION

Investigating the response characteristics of open-channel irrigation-water supply and distribution systems is essential if a good design is to be developed. Numerical simulation techniques have been developed to study the behavior of such systems. A succession of systematically varied simulations can serve to find an optimum design or management procedure.

The set of equations solved in the course of a simulation is a mathematical approximation to the physical laws governing water behavior in nature. Solutions of the equations can provide accurate simulation of natural conditions if the necessary mathematical terms are included in the equations and if the numerical data comprising the site-specific characteristics of the simulation are chosen well. In some cases, it is possible to use only a few terms and still simulate the essential features of the flow.

Equations that fully describe flow in canal systems are nonlinear; closed-form solutions cannot be obtained except in special cases. Instead, numerical techniques are used to calculate values of depth and discharge that approximate solutions to the equations at discrete times at discrete points in the system. Numerical techniques differ in the form and completeness of the basic equations to be solved and in the method used to discretize them. The techniques differ also in the way numerical parameters of the solution are determined, e.g. specification of the solution nodes, the discrete space and time points at which solution is sought. Mathematical models furthermore differ in the complexity of system geometry they allow (e.g. canal gates, branches, networks) and in the flow phenomena they are designed to simulate (e.g. filling a dry canal, seiching, and stationary or moving hydraulic jumps such as bores and supercritical flow). Finally, the techniques...
differ in the ease and convenience of entering and viewing data input and output.

The art of numerical simulation is being able to choose from among the various techniques the method that produces a sufficiently accurate solution with a minimum of computational effort. The purpose of the present paper is to examine techniques used to simulate the different flows that can occur in a canal system and to explain why some methods are preferred.

**BASIC EQUATIONS**

Two physical principles are generally recognized as governing the flow of water: conservation of mass and conservation of momentum. Flow in a canal system typically consists of long reaches of essentially one-dimensional gradually varied flow separated by short zones of rapidly varied flow. For example, rapidly varied flow occurs at pumping plants, at turnout and control structures, and at a hydraulic jump or bore.

In a cross section of gradually varied flow, the pressure distribution is essentially hydrostatic and the velocity distribution is nearly uniform. The conservation principles in this case lead to the so-called Saint Venant equations of continuity and momentum, expressing the flow variables as functions of distance and time. Cunge et al. (1980) presents two formulations of these equations, an integral and a differential form. The integral form is the more general and from it the differential form can be derived. The integral form does not require that the functions be continuous. Therefore, it can be used to simulate problems where, in the absence of a hydraulic structure, water-surface variation is discontinuous, as with a hydraulic jump or bore wave.

In practice, differential equations are used most frequently as the basis for the numerical simulation. When the partial differential equations are in divergent form

\[
\frac{\partial Q}{\partial t} + \frac{\partial A}{\partial t} + q = 0 \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \ CD-638
the opposite case, say with the classical $f = ma$ form of the Saint Venant equations,

$$V \frac{\partial A}{\partial x} + A \frac{\partial V}{\partial x} + B \frac{\partial y}{\partial t} + q = 0 \quad \text{(4)}$$

$$\frac{1}{g} \frac{\partial V}{\partial t} + \frac{V}{g} \frac{\partial V}{\partial x} + \frac{\partial y}{\partial x} = S_0 - S_f + \frac{(V - u)q}{Ag} \quad \text{(5)}$$

the physical presence of a bore leads to unpredictable results. In these expressions, $V =$ velocity, assumed nearly uniform in a cross section; $B =$ top width of the section at the depth $y; f =$ force; $m =$ mass; $a =$ acceleration.

The shortness of zones of rapidly varied flow allows conservation of mass to be expressed as equality of discharge entering and leaving the zone at any instant, as in steady flow. The length of any such zone storing water stemming from differences in discharge is negligible. The dynamics governing the relationship between depths and discharges depends upon the particular flow geometry and can be very complicated. In some cases the relationship can only be predicted empirically. On the grounds that convective acceleration in these rapidly varied flows is much greater than local acceleration, steady-state relations between depths and discharges are assumed valid for the unsteady flow as well. An exception is the case of long siphons.

In a uniform conduit, convective accelerations are nearly zero. The local accelerations $\frac{\partial V}{\partial t}$, however, depending on their sign either augment or diminish the effects of boundary drag. The general expression for closed-conduit flow, the equivalent of the preceding equations for flow with all boundaries rigid is

$$\frac{\partial H}{\partial x} = -S_f - \frac{\partial V}{\partial t} \quad \text{(6)}$$

This equation is easily integrated over the length of a uniform conduit. In (6), $H =$ total head,

$$H = \frac{p}{\gamma} + z + \frac{V^2}{2g} \quad \text{(7)}$$

where $\gamma =$ unit weight of water; $p/\gamma =$ pressure head; and $z =$ elevation.

The friction slope $S_f$ is expressed in terms of the Chézy $C$,

$$S_f = \frac{Q/Q}{A^2C^2R} \quad \text{(8)}$$

and hydraulic radius $R = A/W$, where $W$ is the wetted perimeter.

$C$ can be expressed in terms of the theoretical Colebrook-White equation, based on a logarithmic velocity distribution in pipes. A version for open channels (Henderson 1966) is

$$C = -5.66\sqrt{g} \log_{10} \left( \frac{k}{12R} + \frac{0.88C}{Rc\sqrt{g}} \right) \quad \text{(9)}$$

where the Reynolds number is $R = 4VR/\nu; \nu =$ kinematic viscosity; and $k =$ equivalent sand roughness, a measure of absolute roughness, with dimensions $L$. The viscous term accounts for low Reynolds number turbulent
flow in relatively smooth channels. The solution to (9) is recursive. Only a few iterations are necessary if a seed value of \( C = 60 \) is used to start the computations.

A similar formulation for \( C \) at the high Reynolds numbers and roughness typically found in open channels (Sayre and Albertson 1961) is

\[
C = 6.06 \sqrt{g} \log_{10} \left( \frac{R}{\chi} \right) \tag{10}
\]

in which \( \chi \) = an absolute roughness, dependent, like \( k \), upon the size distribution, shape and spacing of the roughness elements, and measured in units of length.

In terms of the familiar Manning \( n \), the Chézy \( C \) is expressed

\[
C = \frac{c_u}{n} R^{1/6} \tag{11}
\]

where, \( c_u \) = a units coefficient, equal to 1.0 in the metric SI system, and 1.486 in the English system.

The theoretical advantage of the logarithmic formulas over the Manning expression for conditions of high relative roughness is demonstrated in Fig. 1 ("Floods" 1961). In principle, all of the roughness coefficients should depend on absolute roughness alone, and hence remain constant in a channel for all values of normal depth. Instead, Manning’s \( n \) varies by about 300\%, whereas the sand grain roughness \( k \) varies but 30\%, and \( \chi \) only 15\%. The relatively small variation of the coefficients in the logarithmic formulas demonstrates the advantage of the theoretical Colebrook and White approach over the more empirical Manning formula.

At the other end of the scale, for small relative roughness, the Manning \( n \) can be expected to change also. For very large canals, on the order of 10 m deep and 30 m wide (30 ft by 100 ft), the Manning \( n \) for new concrete increases from the standard value of 0.014 to 0.016.

The independent variables in (1)–(6) are distance along the channel \( x \) and time \( t \). The dependent flow variables describe the geometric and kinematic characteristics of the flow. In a one-dimensional flow, all geometric characteristics are determined for a cross section with given channel geometry, once a single geometric variable, like depth or water-surface elevation, is known. With velocity distribution in a cross section known (typically assumed uniform), a single kinematic variable, like discharge, or flow velocity, together with cross-sectional geometry, determines all kinematic characteristics. From the standpoint of numerical simulation, those dependent variables that exhibit the most gradual variation with distance and time are to be preferred. With the bed scour and deposition encountered in alluvial rivers, water-surface elevation is often chosen for the geometric variable, because it varies less than depth. With canals having rigid boundaries, depth is the more commonly used variable. Similarly, Cunge et al. (1980) points out that in steep channels, depth is the preferred variable.

The choice of velocity or discharge as the primary kinematic variable depends upon the rate of change of the respective variables. The discharge in the vicinity of a gate structure or upstream from a free overfall, for example, is virtually constant with distance, while velocity can vary greatly. On the other hand, in a canal-filling operation, near the front of a surge of water advancing on a dry bed, the velocity variation with distance is much smaller than discharge variation.
In any event, equations relating depth and discharge at the boundaries of zones of rapidly varied flow are coupled to the equations relating time and distance variations in depth and discharge in the reaches of gradually varied flow to yield the set of equations governing the total flow in the system. With given initial values of depth and discharge throughout the system and given changes in flow variables as appropriate at the boundaries of the system, the consequent time and distance variation in depth and discharge throughout the system can be determined.
ATTRIBUTES OF NUMERICAL SOLUTION TECHNIQUES

The most important attributes of a simulation technique are applicability, accuracy, and convenience. Applicability refers to the range of system geometries and flow regimes that the technique is designed to handle. Examples of system geometry are gates, weirs, pumps, canal branches, and looped networks. Examples of flow regimes are backwater effects, entry of water into a dry channel, dewatering a sloping channel with upstream inflow cut off, flow overtopping gates, a hydraulic jump downstream of a freeflowing gate, supercritical flow, and bores. A study of the solution algorithms from the point of view of the underlying theory in each case is the best test for applicability. Agreement with theoretical test cases is desirable. If a technique is not designed to handle a specific flow circumstance, the procedure should be capable of detecting the flow condition and stopping the computations with an error message. Otherwise, the user will not know when the results are reliable.

Accuracy in a model means that the values of depth and discharge it predicts are sufficiently close to physical reality to be useful. Simple, theoretical test cases can be devised. For example, the model if allowed to run indefinitely with constant and equal inflow and outflow must converge to a steady-state solution, without manufacturing or losing water. As another example, the water in a channel if allowed to come to rest must exhibit a horizontal water surface. In this regard, the gravitational (slope) and pressure (depth gradient) terms in a model should be so formed that they cancel exactly in a channel of arbitrary shape in still water. This criterion is a major reason for modifying (2) to (3). Ultimately, the best test of accuracy for a theoretically sound model run within its range of applicability is agreement with field and/or laboratory tests.

Convenience refers to the degree of user intervention needed to achieve a solution. For example, adjustment of numerical parameters should not be needed to prevent a program from aborting under developing flow conditions. Associated with convenience is the attribute of robustness. This term has come to mean different things to different people, and the authors propose to resolve current ambiguity by recommending the following unique definition. A robust technique is one that does not produce sawtooth fluctuations in the numerical results, even under severe flow conditions. Typical severe conditions that cause many fragile techniques to abort are shallow depths, high velocities, steep slopes, and rapid changes in discharge. This test for robustness applies to flows that are qualitatively the same, for example, the simulation of subcritical flow at widely varying depths. A technique that breaks down because of a qualitative change in flow regime—such as the development of bores, or supercritical flow—does not necessarily exhibit a lack of robustness. It may merely be inapplicable to the new conditions.

Fragile techniques are likely to develop sawtooth profiles and hydrographs. A sawtoothed solution for depth is likely to drop below zero at some point, usually resulting in an aborted simulation, since depth appears in the equations typically to a non-integer or negative power.

Robustness of a numerical technique is distinct from instability. The latter is defined as inevitable growth without bound of numerical errors in the solution as simulation time progresses, typically from a poor choice of solution nodes or discretization scheme. Many classical causes of instability have been found, by linear analyses of the discretized equations (e.g. Cunge
et al. 1980), and the numerical schemes known to be unstable are routinely avoided.

Robustness is an important issue. Legitimate and relatively nonlegitimate ways exist to achieve robustness. For example, to achieve robustness with shallow flows, the average value of friction slope between node points should be computed with the averages of depth and discharge at the nodes, rather than as the average of friction slopes at the nodes. As suggested earlier, equations with dependent variables that change relatively little and gradually, provide robustness to the numerical scheme used for solution.

Nonlegitimate techniques achieve robustness at the expense of accuracy. For example, introducing numerical overdamping or inappropriate profile smoothing (dissipative interface) improves robustness, but can result in a 5–15% error. Errors up to 100% or more, can be introduced by using an inappropriate but robust model.

The Saint Venant equation of motion relates the forces acting on a cross-sectional slice of water to the acceleration of the water therein. The forces stem from gravity, if the bottom slopes, from the drag of the walls upon the flow, and from the pressure differences arising from a depth gradient. The acceleration terms are a source of fragility, especially if the flow is anywhere near critical; their truncation from the equation (equivalent to assuming all forces in equilibrium) leads to a more robust scheme, labeled zero-inertia, or diffusion (because the governing partial differential equations are now of parabolic type, rather than hyperbolic). This approach has advantages in irregular natural channels, but may be of marginal applicability in prismatic canals. In gate-controlled systems with rapidly changing outflows at turnouts, neglect of acceleration terms can lead to significant inaccuracies and totally precludes simulating the long-period oscillations known as seiching.

Neglect of both the depth-gradient and acceleration terms (tantamount to assuming equilibrium between gravitational and drag forces), leads to the kinematic-wave model. Unless the channel is quite steep, the implied assumption of normal depth everywhere can lead to significant errors. Use of the kinematic-wave model in a channel of small slope is robust, but totally inappropriate. Furthermore, simulation of backwater effects, from operation of downstream gates for example, is precluded with this model.

Robust flood-routing techniques such as the Muskingum and modified Puls methods can be used to model certain classes of unsteady flow in open channels, especially if the coefficients are derived theoretically, instead of empirically. The Cunge (1969) solution for the Muskingum coefficients is based on the zero-inertia version of the equation of motion, but constitutes in addition a perturbation about normal depth. As a consequence, the result is something like an attenuating kinematic wave, incapable of modeling backwater effects. The exceptionally robust modified Puls method approaches, with a very large number of subreaches, the kinematic-wave method. In general, the results depend in a primary way on the size of the subreaches, and depending on the choice of the latter, can yield large errors on either side of the true flow values. Thus, the method is not recommended for general use.

Finally, the techniques should be implemented in computer code that executes quickly, has small memory requirements, uses modular construction, and provides convenient input and output. These attributes allow the program to be used on microcomputers or within larger design programs and provide for ease in trouble shooting or modification.
NUMERICAL SOLUTION METHODS

As in any numerical method used to solve equations describing continuous variation of variables in a continuous medium, the issue of interpolation, and hence, numerical approximation to derivatives or integrals arises. The question of whether a function is assumed to be linear, or of second or higher degree between node points, is a central issue in numerical analysis. Since the Saint Venant equations constitute a hyperbolic system of partial differential equations, a more pressing issue must be decided first—whether or not to deal with the equations in characteristic form.

Method of Characteristics

The Saint Venant equations express variations of flow variables with distance x and time t. When a hyperbolic system of equations is transformed to orthogonal directions \((s, n)\) at some angle to the \((x, t)\) directions, two angles can be found for which the terms containing the partial derivatives in the \(n\)-direction become proportional. Then a linear combination of the two equations allows those terms to drop out, leaving an ordinary differential equation for each \(s\)-direction.

With reference to Fig. 2, a formal transformation of (1) from independent variables \(x, t\) to \(s, n\), leads to the results

\[
\frac{\partial Q}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial Q}{\partial n} \frac{\partial n}{\partial x} + \frac{\partial A}{\partial s} \frac{\partial s}{\partial t} + \frac{\partial A}{\partial n} \frac{\partial n}{\partial t} + q = 0 \quad \text{(12)}
\]

or

\[
\frac{\partial Q}{\partial s} \cos \alpha - \frac{\partial Q}{\partial n} \sin \alpha + \frac{\partial A}{\partial s} \sin \alpha + \frac{\partial A}{\partial n} \cos \alpha + q = 0 \quad \text{(13)}
\]

A suitable choice of \(\alpha\) can yield any desired ratio between the terms containing the \(n\)-derivative of \(Q\) and those containing the \(n\)-derivative of \(A\). A similar transformation of (2) or (3) shows that the ratio of terms containing the \(n\)-derivative of \(Q\) and those containing the \(n\)-derivative of \(A\)

FIG. 2. Orthogonal \(s-n\) Coordinate System Inclined at Angle to \(x-t\) Coordinate System
also depends upon $c$. The appropriate value of $c$ to make those two ratios the same depends upon the values of the flow variables $y$ and $Q$ at the given $x$, $t$. With the two ratios made identical, the continuity equation can be added to a factor times the motion equation to make the terms containing the normal derivatives drop out (in practice, these operations are carried out by straightforward matrix multiplication). What remains is an ordinary differential equation, i.e. containing derivatives in $s$ only, accompanied by another differential equation giving the direction $\alpha$. This form is known as the characteristic form of the original equations. The original set of equations is defined as hyperbolic, because, as a result of their structure, two distinct, real characteristic directions $\alpha$ can be found for each point of the $x$-$t$ plane (Garabedian 1964). A point of zero depth is an exception—the two directions become the same; at such a point special treatment is required.

Following one additional transformation, that in the $s$-direction, $\frac{d}{ds} = \sin \alpha \frac{d}{dt}$, the four ordinary differential equations comprising the characteristic form of the Saint Venant equations can be written

$$\begin{align*}
\frac{dV}{dt} &+ \frac{g}{c} \frac{dy}{dt} = g(S_0 - S_f) + \frac{cVA_y^x}{A} - q \frac{u - V \pm c}{A} \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (14)
\end{align*}$$

each, respectively, valid on

$$\frac{dx}{dt} = V \pm c \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (15)$$

Here

$$c = \pm \sqrt{\frac{gA}{B}} \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (16)$$

As noted, the two $s$-directions are themselves defined by ordinary differential equations, the solutions of which can be plotted in the $x$-$t$ plane as two intersecting families of curves, called, simply, characteristics. Simultaneous numerical solution of all four ordinary differential equations allows the simulation to progress in the long reaches of gradually varied flow, from known conditions to new conditions. Indeed, if depth and discharge are known at some point in the $x$-$t$ plane, say $L$ in Fig. 3, an approximate value for the right-hand side of the first of (14) is known. This implies a known relation between changes in $y$ and changes in $V$ in the direction specified by the first of (15). Neither one can be found individually without the assistance of some additional relationship. This is provided by the second of (14), written for an adjacent known point in the $x$-$t$ plane, $R$ in Fig. 3, which results in a relation between changes in $y$ and $V$ in the direction given by the second of (15). Simultaneous solution at point $P$, where the characteristics emanating from $L$ and $R$ intersect, yields both $y$ and $Q$ at that point, now a known point.

At boundaries of the reaches of gradually varied flow, the pair of equations for one of the characteristics is solved in conjunction with the (algebraic) equations describing the flow in the rapidly varied zone, or known boundary conditions.

The physical significance of the transformation to ordinary differential equations lies in the fact that solutions of ordinary differential equations are dependent upon initial conditions at a starting point. Consider, in Fig. 4, that at time 0 (the $x$-axis), the flow is steady, and that conditions at the
upstream end of the channel (the $t$-axis) are changed, say at $G$, disturbing the initial steady state. The flow lying under curve $GJ$ in Fig. 4 is in the undisturbed steady state. Above $GJ$, the solution reflects the change introduced at $G$. Thus, the physical disturbance introduced at $G$ travels downstream at a velocity equal to the inverse slope of the characteristic $GJ$ emanating from $G$. The characteristics reflect the wave character of unsteady flow in open channels.

If a network of continuous characteristic curves is constructed in the course of the solution (e.g. Fig. 5), the result is as close as possible to theoretical solution of the Saint Venant equations, exhibiting, strictly, the inherent wave character of the flow. Furthermore, the characteristics often draw close together in regions where the flow variables are changing most rapidly, and this is conducive to an accurate numerical solution. A network of characteristic curves, however, is inconvenient, because the intersections
of the two families of curves, where the values of depth and discharge are determined, fall at irregularly located points in distance and time. This is a troublesome, but not unsurmountable problem, when the characteristic equations must be coupled to hydraulic-structure equations at fixed locations.

In an alternate method, the $x$-$t$ plane is covered with a rectangular grid in which lines of constant time and distance are specified. Segments of characteristics are constructed from a time at which values are known to a node on a subsequent time line (Fig. 6). For stability of the successive computations, the depth and discharge values at the beginning of the characteristic segment must be found by interpolation between computed nodes. If the interpolation nodes are always taken one distance step upstream and downstream from the subject node, this is equivalent to the well known Courant condition limiting time steps relative to distance steps. That is, the Courant number, (19), is limited to values less than or equal to unity.

Because a fixed-grid network deals only with segments of the characteristic curves, a very gradual approach and eventual intersection of a neigh-
boring pair from the same family will not be apparent. The intersection of characteristics from the same family is the sign of bore formation. As a rough test, the possibility of an intersection within a time step of two characteristic segments of the same family coming from neighboring node points can be investigated. This test can detect the larger, more suddenly formed bores, but not necessarily those that form gradually in the interior of a reach. Attempts to ignore bore formation when physically one exists lead to unpredictable results.

In a compromise technique, one family of characteristics is allowed to remain continuous, while the second is segmented, the intersections falling on given lines at fixed $x$ or $t$. Then the model can specify either fixed distance lines, or fixed time lines. The version of Fig. 7 (Sivaloganathan 1978) is particularly appropriate for flow in controlled canal systems, with the structures located at fixed $x$. Interpolation is then performed with respect to $t$. However, interpolation along a time line, as in Fig. 7, becomes ever more problematical as the Froude number approaches unity.

Maintaining one family continuous does not guarantee capturing an intersection. For example, a continuous family of characteristics pointed downstream will not exhibit an intersection resulting from increases in discharge upstream if the channel is not long enough for the intersection to

![Diagram](image-url)

**FIG. 7. Method of Characteristics—One Family Continuous**
take place within its length. The intersection may well occur between a pair of characteristics pointed upstream, reflected at the downstream boundary from a converging pair of characteristics initially pointed downstream (Strelkoff 1992).

At the differential level, the Saint Venant equations and the characteristic equations are equivalent. Solutions numerically integrated over finite steps on characteristic curves, however, have been demonstrated to violate mass conservation (and one expects, also, momentum conservation), with the system either emptying or filling, even under constant and equal inflow and outflow. This is especially true with the rectangular grid. Second-degree (as opposed to linear) interpolation and integration formulas, small time and distance steps, and, in the rectangular net, ratios of time and distance steps close to the slope of the characteristics all reduce this error. Furthermore, if the flow variables are changing more slowly with time than with distance, interpolation along a time line will yield more accurate results than interpolation along a distance line.

Characteristics-based methods are equally applicable to subcritical and supercritical flow. They alone can demonstrate bore formation, but cannot be used in themselves to calculate its size or movement. The bore must be isolated, and the algebraic mass- and momentum-conservation equations governing the relationship between depths and discharges on its high and low sides and velocity of propagation $W_B$

$$W_B = \frac{Q_L - Q_R}{A_L - A_R}$$ ................................................ (17)

$$V_L - V_R = + \sqrt{\frac{g(P_L - P_R)(A_L - A_R)}{A_LA_R}}$$ ........................................ (18)

must be coupled to a pair of characteristic equations on the high side. On the low side, the two pairs of characteristic equations alone are sufficient to determine depth and discharge. In (17) and (18), the subscripts $L$ and $R$ refer to the left and right sides of the shock, respectively, irrespective of which is the high side (the $x$-coordinate is assumed increasing from left to right). The positive sign is chosen for the radical in (18) to ensure that water always crosses the bore from the low side to the high side and so lose energy rather than gain it, i.e., $(V_L - W_B) > (V_R - W_B)$ must hold.

Only the theory of characteristics provides the basis for determining the necessary and sufficient initial and boundary conditions to simulate an unsteady open-channel flow. This information, however, must be used with all solution methods.

Very shallow flows, as at the leading and trailing edges of a wave on a dry bed (e.g. canal filling or dewatering), are not convenient to calculate by characteristics, because the two families of characteristics intersect at such a small angle, the location of the intersection presents a poorly posed problem—slight differences in input data or solution precision can lead to large differences in results.

**Numerical Solution of Saint Venant Equations in x-t Coordinates**

In characteristics-based methods, the governing equations are integrated in directions $s$ sloping at angles to the $x$ and $t$ directions. An alternative approach, popular because of its intuitive simplicity and the regular spacing of the solution grid points, solves the Saint Venant equations in their original
noncharacteristic form, with $x$ and $t$ as the independent variables. Even though the noncharacteristic, conservative form of the Saint Venant equations can easily be made to satisfy volume conservation exactly even with a coarse net, the results for depth and discharge are generally somewhat damped and distorted, relative to solution of the characteristic equations. Finite-difference techniques approximate the partial derivatives in the Saint Venant equations with quotients of finite differences between node points. Integral techniques approximate by quadratures the Saint Venant equations integrated over finite time steps and over cells of water comprising in toto the length of the subject reach. Liggett and Cunge (1975) and Cunge et al. (1980) discuss a number of different schemes.

First-order schemes assume a linear variation of solution variables between grid points; second-order schemes use a second-degree variation, and so forth. Theoretically, as a grid is made finer and finer, second-order schemes approach the theoretical solution faster than first-order schemes once the grid elements are sufficiently small. In practice, the grid elements are often not small enough for this criterion to be of great importance, relative to other factors governing solution accuracy.

Once the solution network departs from the characteristics grid, inaccuracies arise that cannot be reduced simply by reducing the grid spacing. For any given discretization scheme, they are dependent upon the ratio of time and distance steps and upon the rapidity of change in the boundary conditions. Three concerns need to be addressed: instability, the growth without bound of inevitable numerical errors of truncation and roundoff; numerical damping, the reduction of peaks and filling of troughs in the solution; and numerical dispersion, the development of high-frequency waviness in the solution, when the physical wave motion exhibits only long-period variations. The nonlinearity of the Saint Venant equations precludes significant theoretical investigation of their numerical solution. Much can be learned, however, from analysis of the linearized versions, in which solutions to the discretized linear equations are compared to analytic solutions of the linearized differential equations. The linearization of the equations permits superposition of solutions. In particular, solutions are viewed as Fourier series, with components at wavelengths dependent upon the grid spacing.

Instability is studied by examining the amplification of the solution-component amplitudes from one time step to the next. Numerical damping and dispersion are viewed in terms of amplitude and phase portraits, respectively. The amplitude portrait is a plot of numerical damping as a function of solution-component wavelength, and the phase portrait is a plot of numerical wave celerity as a function of wavelength. Typically, damping and celerity are not uniform over the spectrum of wavelengths; this leads to distortion of the computed wave profiles. An important parameter of the numerical solution is the ratio of time-step size $\Delta t$ to distance-step size $\Delta x$, relative to the magnitude of the inverse slope of the characteristic curves. This can be expressed in terms of the Courant number

$$C_r = \frac{|V| + c}{\Delta x} \Delta t$$ ................................. (19)

While the results of linear analysis cannot be used for quantitative prediction in the nonlinear case, it appears that qualitatively, the schemes usually behave as expected, and the analyses are useful in comparing the different approaches.
Two methods are used to solve the equations: explicit and implicit. Explicit methods solve the governing equations at one point on the time line of unknowns, in terms of known values. The solution in implicit methods is found simultaneously for all points on the time line of unknowns. Integral methods are typically implicit. As compared to explicit schemes, the large number of equations to be solved simultaneously in the implicit schemes requires more complex solution algorithms; these, in turn, are more difficult to troubleshoot. The implicit schemes are, however, free of stability constraints on time-step size relative to distance step. Accuracy requirements, however, can still dictate the use of small time or distance steps. In particular, a flow with steep x-gradients will require closely spaced node points in distance. An explicit scheme would require correspondingly small time steps (for stability, the Courant condition, \( C_r \leq 1 \), must be satisfied); an implicit scheme would be free of this linkage. If the steep gradients are the result of rapid time changes, as in emergency canal operation, then a rectangular grid requires small time steps as well. An oblique grid following the wave would be free of this requirement. In any event, the issue is one of accuracy of approximation and robustness, rather than stability.

Implicit schemes for subcritical flow, solved with a boundary condition at each end, inherently contradict the wave nature of unsteady flow. This specifies that boundary conditions affect conditions in the interior only some time after they are instituted, when the characteristic reflecting the change, emanating from the boundary, reaches the point in question. The grid-cell numerical equations for the entire reach, on the other hand, are solved simultaneously, and so interact, at an instant of time. This theoretical anomaly, however, has not, in practice, overcome the many advantages of the implicit formulation.

Fig. 8 illustrates a simple explicit finite-difference scheme, designed for a fixed network in the \( x-t \) plane. Time derivatives are approximated by the forward differences between \( P \) and \( M \); distance derivatives by central differences between \( L \) and \( R \). Stability and damping are influenced by the parameter \( \alpha \) in the expression of conditions at \( M \)

\[
f(M) = \alpha f_{Mi} + (1 - \alpha) \frac{f_R + f_L}{2} \quad \text{..........................} \quad (20)
\]

in which \( f \) = any solution variable; and the subscript \( i \) refers to the solution.
obtained at the corresponding points in the course of the preceding time step. The value, \( \alpha = 1 \), leads unconditionally to instability. With conditions at \( M \) found solely by interpolation (\( \alpha = 0 \)), the solutions are heavily damped and diffuse. Stability is provided if the Courant number is less than unity. In general, the scheme is relatively robust, but introduces significant numerical diffusion and damping.

The implicit scheme of Fig. 9 is not bound by the Courant condition required by Fig. 8, but it is also diffusive and numerically damped. The leap-frog scheme of Fig. 10, is second-order and more accurate than the preceding two, but it depends upon the Courant condition for stability.

All of the previous schemes share the complication that at boundaries of the gradually varied flow zones, they must be solved in conjunction with the numerical approximation to an appropriate characteristic equation. The first of (14) applies for a downstream boundary and the second for an upstream boundary. This restriction is absent from the box scheme of Fig. 11 (Preissmann 1961; Amein and Fang 1970). Time derivatives for the grid

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**FIG. 9. Implicit (T) Scheme**

**FIG. 10. TVA Explicit (Leap-Frog) Scheme**
cell are the numerical averages of time derivatives on the right \((MR)\) and left \((JL)\) sides. Spatial derivatives are a weighted average of derivatives on the time lines of unknowns \((LR\), weighted by a factor \(\theta\)) and knowns \((JM\), weighted by \([1-\theta]\)). Cell averages for coefficients and constant terms are similarly weighted. For stability, the weighting factor must be at least equal to 1/2. The value of unity leads to well behaved, heavily damped results. A compromise value is taken, typically, just over the half-way point, 

\[
0.55 \leq \theta \leq 0.6
\]

Written in momentum-conservation terms, derived from (1) and (3), this model is becoming the standard approach for commercial models.

Integral methods apply the volume- and time-integrated conservation equations to the water cells between distance nodes over finite increments of time. The integrals are approximated in terms of nodal values of the integrands by quadratures, such as the trapezoidal rule. Because integrals of irregular functions are easier to accurately express numerically than derivatives, even with very small steps, one would expect integral methods to be more robust than finite-difference methods. However finite-difference methods, possibly for historical reasons, are far more common.

In general, flows with steep positive waves are simulated most accurately and smoothly with oblique networks in the \(x-t\) plane, in which the moving steep wave front is contained within a group of node points that move along the channel with time, as in Fig. 12. The numerical advantages of such a moving grid are reduced if the channel contains gates or other physical features that require some node points to be fixed in location. While the integral (and integral-derived) methods allow computation of bores without isolating them, to provide robustness, large bores still require tracking. The conservation equations for gradually varied flow with and without a bore are identical only if the bore is centered between distance nodes in a rectangular channel (Terzidis and Strelkoff 1970), near the center of a trapezoidal channel, or at a node itself. Steep wave fronts that are not bores are most likely in a canal-filling operation. The zero depth at such a front poses a problem for solutions of the unsteady-flow equations. A plausible physical
assumption for dealing with such a wave front is that the velocity varies but little with distance back from the leading edge (Whitham 1955). This leads to a profile shape for the tip of the wave that can be coupled to the main body, computed by standard means.

Supercritical flow is often encountered in controlled canals under low-flow conditions. An example is the jet from under a partially open control gate that discharges without submergence and connects to the downstream flow through a hydraulic jump. The phenomenon is particularly difficult to simulate if the gate is followed by a siphon where the jump can form upstream from, downstream from, or within the siphon. It is also possible that the location of the jump will appear as a transitory, numerical phenomenon, in the course of iterations for the finally accepted result at a given time. Present numerical schemes are not geared to handle such circumstances. Introduction of supercritical flow into schemes not designed to handle them leads to unpredictable oscillatory results. In particular, implicit schemes in which the simultaneous equations are solved with a double-sweep method—having one boundary condition picked up at the upstream end of a canal and another picked up at the downstream end—exhibit oscillations if the Froude number exceeds unity by more than just a little or for more than just a short time. The problem here is solely with the method of solving the algebraic simulation equations, not with the equations themselves.

**PRACTICAL CONSIDERATIONS**

If a given scheme is applicable to the geometry and flow conditions to be simulated, is stable, and has convenient data entry and output, it can be further evaluated with a series of test runs.

Long-term steady-state simulation can disclose unreasonable violation of mass conservation. Furthermore, it should be possible to decrease the time and distance steps to a point at which further decreases lead to no further change in results. Frequent output of the volume error generated by the program during the simulation is desirable. Postulation of hypothetical, severe flows can disclose fragility. Conveniently obtained profile plots during a simulation are helpful in evaluating the progress of the solution. If the
plots are displayed on the computer screen, the operation that causes a difficulty in the simulation can sometimes be quickly identified.

Physical test cases should be simulated to test the model for accuracy. One of the most difficult factors to evaluate is the boundary friction characteristic. On the grounds that boundary resistance is a function of relative roughness, the Manning n can be expected to change with changes in canal size or depth. The Colebrook-White formula overcomes much of the weakness of the Manning formula and is, fortunately, in increasing use in the United States and internationally. Nevertheless, factors such as weed, algae, and barnacle growth are difficult to evaluate.

Numerical analysis is a developing field, and computer programs built to simulate unsteady flow in open channels are becoming increasingly available. Newer programs are particularly attractive in their user-friendliness, with menu-driven data entry, and sophisticated color graphics of hydrographs, profiles, maps of flooded areas, and so forth. Users of these programs often are unaware of the techniques, with their inherent limitations, used in preparing the programs. Because of this, users frequently are unaware of the approximate nature or inapplicability of the numerical solution. The tendency to believe blindly the computer output to be the true solution must be resisted. The results of even the most sophisticated and user-friendly computer program should be treated with skepticism and thoroughly investigated for soundness. If possible, the results should be verified with field data or at least, with carefully formulated test problems.

ACKNOWLEDGMENTS

The writers are grateful to John G. Sakkas for the computations reflected in Fig. 5.

APPENDIX. REFERENCES


