Decomposing growth in revenues and costs into price, quantity and total factor productivity contributions

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This article employs the superlative Fisher and Törnqvist indexes for exact decomposition of growth in nominal revenues and costs. The findings confirm the well-known result that these indexes very closely approximate each other, implying that the mathematically simpler and computationally easier Törnqvist is the more practicable index. Moreover, this article's nominal growth decomposition yields all the results from the more common real growth decomposition and is also more informative for policy purposes. Application to the US agricultural sector during 1948–2001 shows that of the 3.31\% average annual growth in revenues, TFP growth contributed 1.90 percentage points (pct. pts.); growth in output prices added 1.43 pct. pts.; while growth in input quantities contributed – 0.02 pct. pts. (i.e. fewer inputs). Therefore, real output growth (or revenue growth less output price growth) was 1.88 pct. pts., revealing that TFP’s 1.90 pct. pts. growth contribution was fully responsible for real output growth with fewer inputs. Since revenues measure incomes, these results suggest that policy should focus more on measures to foster TFP growth than on specific price or quantity instruments to enhance income growth.

I. Introduction

This article proposes a nominal growth decomposition framework that yields all the results from the common procedure focusing on real growth decomposition. Thus, the proposed decomposition framework is analytically more general. It is also more informative for policy purposes. Since revenues or costs measure incomes, our growth decomposition determines the contributions of prices, quantities and total factor productivity (TFP) to income growth. The results have important implications for income-enhancing policies because policies affecting prices or quantities could differ from those designed to boost productivity growth.

Our strategy is to examine price and quantity indexes jointly, specifically examining changes in the product of these indexes. Considering that an

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index is conceptually a ratio (i.e. measures relative change), the product of the output price and quantity indexes measures the relative change in nominal revenues while the product of the input price and quantity indexes measures the relative change in nominal costs. In the case of the Fisher index, the measure is exact from the fact that the product of a Fisher quantity index and of the corresponding Fisher price index exactly equals the relative change in nominal value, which is the well-known factor reversal property (Fisher, 1922). In contrast, the product of a Törnqvist quantity index and of the corresponding Törnqvist price index approximately equals the relative change in nominal value, i.e. the Törnqvist lacks the factor reversal property. However, this lack is empirically inconsequential as evidenced by our findings that the results from the Fisher and Törnqvist growth decompositions are indistinguishable when carried out at the usual level of precision. Henceforth, for brevity, we drop ‘nominal’ when referring to revenues and costs.

Our basic framework is an index number representation of the revenue function and of the cost function so that our growth decomposition is consistent with standard economic theory. The framework is practical because it utilizes either the Fisher index or the Törnqvist index that is easily constructed from available data. It is also theoretically rigorous because these indexes are ‘superlative’ (Diewert, 1976).

We show that an exact decomposition of growth in revenues or costs is possible based on the Fisher index. Diewert and Morrison (1986) derived an exact decomposition of revenue growth based on the Törnqvist index. This decomposition was implemented by Kohli (1990) and by Gopinath and Roe (1996, 1997), and was adapted by Kohli (2003) in decomposing growth in costs. However, as noted above, the Fisher index satisfies factor reversal but not the Törnqvist index. This gives an analytic advantage to the former index over the latter. This advantage is that, assuming a constant returns to scale technology, the Fisher TFP index in the revenue side equals the Fisher TFP index in the cost side. In contrast, the Törnqvist revenue-side TFP index only approximates – although very closely – the Törnqvist cost-side TFP index.

We employ a reformulation (Reinsdorf et al., 2002; Kohli, 2003; Balk, 2004) of the Fisher index that looks mathematically similar to the Törnqvist. This reformulation makes these indexes transparently comparable. Our comparison shows that the Törnqvist is simpler than the Fisher and clearly pinpoints the sources of differences between the growth rates of these indexes. The analytical and empirical findings complement earlier results (Diewert, 1976, 1978; Dumagan, 2002) that these indexes are close approximations to each other.

Our findings show that the Fisher and Törnqvist growth decomposition procedures are analytically very similar and yield virtually the same empirical results. These findings imply that the mathematically simpler and computationally easier Törnqvist growth decomposition for the revenue side (Diewert and Morrison, 1986; Kohli, 1990; Gopinath and Roe, 1996, 1997) and for the cost side (Kohli, 2003) is more practicable than the Fisher procedure.

In our application, we use both the Fisher and Törnqvist indexes to decompose the growth in revenues of the US agricultural sector into the contributions of growth in output prices, input quantities and TFP. We assume that the data are generated by a constant returns to scale technology so that factor payments (including an imputation for self-employed labour) just exhaust the value of output. This result allows a second decomposition of the growth in total costs of the sector into the contributions of output quantities, input prices and TFP.

II. Superlative Index Decompositions of Growth in Revenues and Costs

Let time change from period $s$ to period $t$ and let there be $M$ outputs, $i = 1, 2, \ldots, M$ and $N$ inputs, $j = 1, 2, \ldots, N$, in each period. In these two periods, the output price vectors are $P_i = \{p_{i,s}\}$ and $P_t = \{p_{i,t}\}$.
and the corresponding output quantity vectors are \(Q_s = \{q_{is}\}\) and \(Q_t = \{q_{it}\}\). Similarly, the input price vectors are \(W_s = \{w_{is}\}\) and \(W_t = \{w_{it}\}\) and the input quantity vectors are \(X_s = \{x_{is}\}\) and \(X_t = \{x_{it}\}\).

Using dot products of the above price and quantity vectors, the relative changes from \(s\) to \(t\) in revenues \((R_{st})\) and in costs \((C_{st})\) are, by definition,

\[
R_{st} = \frac{P_{st} \cdot Q_{st}}{P_s \cdot Q_s}; \quad C_{st} = \frac{W_{st} \cdot X_{st}}{W_s \cdot X_s} \tag{1}
\]

In turn, \(R_{st}\) and \(C_{st}\) can be expressed exactly in terms of superlative index number formulas.

As a preliminary, the Laspeyres and Paasche indexes can be denoted by superscript \(L\) and \(P\) indexes for output prices \((P)\) and quantities \((Q)\) are,

\[
P^L_{st} = \frac{P_{st} \cdot Q_{s}}{P_s \cdot Q_s}; \quad P^P_{st} = \frac{P_{st} \cdot Q_{t}}{P_t \cdot Q_t}; \quad Q^L_{st} = \frac{Q_{s} \cdot P_{s}}{Q_s \cdot P_s}; \quad Q^P_{st} = \frac{Q_{t} \cdot P_{t}}{Q_t \cdot P_t} \tag{2}
\]

Similarly, the Laspeyres and Paasche indexes for input prices \((W)\) and quantities \((X)\) are,

\[
W^L_{st} = \frac{W_{st} \cdot X_{s}}{W_s \cdot X_s}; \quad W^P_{st} = \frac{W_{st} \cdot X_{t}}{W_t \cdot X_t}; \quad X^L_{st} = \frac{X_{s} \cdot W_{s}}{X_s \cdot W_s}; \quad X^P_{st} = \frac{X_{t} \cdot W_{t}}{X_t \cdot W_t} \tag{3}
\]

The formulas for the superlative Fisher and Törnqvist indexes are presented below.

**Fisher index decomposition of growth in revenues and costs**

The Fisher indexes (denoted by superscript \(F\)) are, by definition, the geometric means of the corresponding Laspeyres and Paasche indexes. Reinsdorf et al. (2002), Kohli (2003) and Balk (2004) showed that the Laspeyres and Paasche indexes can be expressed as the weighted geometric means of the individual price or quantity relatives. As a result, the Fisher output price and quantity indexes can be expressed as,

\[
P^F_{st} = (P^L_{st} P^P_{st})^{1/2} = \prod_{i}^{M} \left( \frac{p_{st}}{p_{is}} \right)^{1/2} \left( \frac{p_{st}}{p_{it}} \right)^{1/2} \tag{4}
\]

\[
Q^F_{st} = (Q^L_{st} Q^P_{st})^{1/2} = \prod_{i}^{M} \left( \frac{q_{is}}{q_{is}} \right)^{1/2} \left( \frac{q_{it}}{q_{it}} \right)^{1/2} \tag{4}
\]

The expressions in Equation 4 are the multiplicative decompositions of the Fisher indexes. The exponents or weights in the output quantity index in Equation 4 are given by,

\[
\lambda^L_{s,t} = \frac{\alpha^{L,q}_{s,t}}{\sum_{i} \alpha^{L,q}_{s,t}}; \quad \alpha^{L,q}_{s,t} = \frac{q_{is}/q_{it} - Q^L_{s,t}}{\ln(q_{is}/q_{it}) - \ln(Q^L_{s,t})}; \tag{5}
\]

\[
\lambda^P_{s,t} = \frac{\alpha^{P,q}_{s,t}}{\sum_{i} \alpha^{P,q}_{s,t}}; \quad \alpha^{P,q}_{s,t} = \frac{1/(q_{is}/q_{it}) - 1/Q^P_{s,t}}{\ln(1/(q_{is}/q_{it}) - \ln(1/(Q^P_{s,t}))}; \tag{6}
\]

\[
\sum_{i} \lambda^L_{s,t} = 1; \quad \sum_{i} \lambda^P_{s,t} = 1; \tag{7}
\]

In Equations 5 and 6, \(r_{is}\) and \(r_{it}\) are revenue shares. To obtain from Equations 5 and 6 the weights \(\lambda^L_{s,t}\) and \(\lambda^P_{s,t}\) for the output price index in Equation 4, replace \(q_{is}\) and \(q_{it}\) by \(p_{is}\) and \(p_{it}\) and also replace \(Q^L_{s,t}\) and \(Q^P_{s,t}\) by \(P^L_{s,t}\) and \(P^P_{s,t}\).

By similar procedures, the Fisher input price and quantity indexes become,

\[
W^F_{st} = (W^L_{st} W^P_{st})^{1/2} = \prod_{j}^{N} \left( \frac{w_{st}}{w_{js}} \right)^{1/2} \left( \frac{w_{st}}{w_{jt}} \right)^{1/2} \tag{8}
\]

\[
X^F_{st} = (X^L_{st} X^P_{st})^{1/2} = \prod_{j}^{N} \left( \frac{x_{st}}{x_{js}} \right)^{1/2} \left( \frac{x_{st}}{x_{jt}} \right)^{1/2} \tag{8}
\]

The weights for the input quantity index in Equation 8 are given by,

\[
\lambda^L_{s,t} = \frac{\alpha^{L,x}_{s,t}}{\sum_{j} \alpha^{L,x}_{s,t}}; \quad \alpha^{L,x}_{s,t} = \frac{x_{st}/x_{js} - X^L_{st}}{\ln(x_{st}/x_{js}) - \ln(X^L_{st})}; \tag{9}
\]

\[
\lambda^L_{s,t} = \frac{w_{st} x_{st}}{w_{is} x_{is}} \tag{9}
\]

5 The identity \((=)\) is used above for definitions and the equality \((=)\) is used for results of algebraic operations.

6 Balk (2004) showed that the Fisher quantity index \(Q^F_{s,t}\) has both additive and multiplicative decompositions. The difference is that the additive decomposition yields the arithmetic growth rate \(\ln(Q^F_{s,t})\) while the multiplicative decomposition yields the logarithmic growth rate \(\ln(Q^F_{s,t})\). Dumagan (2002) derived the exact additive decomposition—that Balk (2004) recognized as a rediscovery of van IJseken’s (1952) decomposition—and showed that the Fisher arithmetic growth rate \((Q^F_{s,t} - 1)\) and the Törnqvist logarithmic growth rate \(\ln(Q^F_{s,t})\) mathematically approximate each other. This present article provides (in a later section) a new analytic basis for the closeness between the logarithmic growth rates \(\ln(Q^F_{s,t})\) and \(\ln(Q^F_{s,t})\) to solidify the earlier finding that the Fisher and Törnqvist indexes approximate each other.
\[
\lambda_{j,t,s}^{P_X} = \frac{\alpha_{j,t,s}^{P_X}}{\sum_j \alpha_{j,t,s}^{P_X}};
\]
\[
\alpha_{j,t,s}^{P_X} = c_{j,t} \left[ \ln \left( \frac{1}{(x_{j,t}/x_{j,s})} \right) - \ln \left( \frac{1}{(X_{j,t}^{P_X})} \right) \right];
\]
\[
c_{j,t} = \frac{W_{j,t}X_{j,t}}{W_i X_i};
\]
\[
\sum_j J \lambda_{j,t,s}^{L_X} = 1; \quad \sum_j J \lambda_{j,t,s}^{P_X} = 1; \quad \sum_j J c_{j,t} = 1; \quad \sum_j J c_{j,t} = 1
\]

(11)

In Equations 9 and 10, \(c_{j,t}\) and \(c_{j,t}\) are cost shares. Obtaining the weights \(\lambda_{j,t,s}^{L_X}\) and \(\lambda_{j,t,s}^{P_X}\) for the input price index in Equation 8 should now be obvious.

By the factor reversal property (see footnote 1), \(R_{st}\) equals the product of the Fisher output price and quantity indexes and \(C_{st}\) equals the product of the Fisher input price and quantity indexes. That is, Equations 1, 2 and 3 yield,

\[
R_{st} = P_{st}^{F}Q_{st}^{F}, \quad C_{st} = W_{st}^{F}X_{st}^{F}
\]

(12)

Moreover, by definition, the Fisher TFP index for the revenue side, \(E_{st}^{FR}\), is the ratio of the output quantity index to the input quantity index. Also by definition, the Fisher TFP index for the cost side, \(E_{st}^{FC}\), is the ratio of the input price index to the output price index. That is,

\[
E_{st}^{FR} = \frac{O_{st}^{F}}{X_{st}^{F}}, \quad E_{st}^{FC} = \frac{W_{st}^{F}}{P_{st}^{F}}
\]

(13)

Diewert (1976, 1992) showed that the Fisher TFP indexes in Equation 13 are equal to the productivity indexes derived under competitive profit-maximizing conditions from flexible ‘quadratic’ revenue and cost functions. This equality means that Equation 13 is exact and, because the revenue and cost functions are flexible, Equation 13 is superlative. Moreover, by assumption, the production function exhibits constant returns to scale. Together with competitive profit maximization, this implies revenues equal costs in each period. Therefore, \(R_{st} = C_{st}\) in Equation 12 and the two Fisher TFP indexes in Equation 13 are equal.

Substituting Equation 13 into Equation 12 yields a Fisher index representation of revenue and cost functions,

\[
R_{st} = P_{st}^{F}X_{st}^{F}E_{st}^{FR}, \quad C_{st} = W_{st}^{F}Q_{st}^{F}E_{st}^{FC}
\]

(14)

The first equation in (14) embodies the properties of a revenue function that revenues rise with a rise in either or both output prices and input quantities (i.e. positive marginal products) and with growth in TFP.7 The second embodies the properties of a cost function that costs rise with a rise in either or both input prices and output quantities (i.e. positive marginal costs) but fall with TFP growth. Hence, TFP growth is revenue-enhancing and, at the same time, it is cost-reducing.

Combining Equations 4, 8 and 14 yields,

\[
R_{st} = \prod_i \left( \frac{P_{i,t}^{L_X}}{P_{i,t}^{F}} \right)^{1/2} \prod_j \left( \frac{X_{j,t}^{L_X}}{X_{j,t}^{F}} \right)^{1/2} \prod_i \left( \frac{W_{i,t}^{L_X}}{W_{i,t}^{F}} \right)^{1/2} \left( \prod_i \left( \frac{P_{i,t}^{L_X}}{P_{i,t}^{F}} \right)^{1/2} \right)^{-1}
\]

(15)

\[
C_{st} = \prod_j \left( \frac{W_{j,t}^{L_X}}{W_{j,t}^{F}} \right)^{1/2} \prod_i \left( \frac{Q_{i,t}^{L_X}}{Q_{i,t}^{F}} \right)^{1/2} \left( \prod_i \left( \frac{W_{i,t}^{L_X}}{W_{i,t}^{F}} \right)^{1/2} \right)^{-1}
\]

(16)

By taking natural logarithms, Equations 15 and 16 yield,

\[
\ln(R_{st}) = \sum_i \frac{1}{2} \left( \lambda_{i,t}^{L_X} + \lambda_{i,t}^{P_X} \right) \ln \left( \frac{P_{i,t}^{L_X}}{P_{i,t}^{F}} \right)
\]

\[
+ \sum_j \frac{1}{2} \left( \lambda_{j,t}^{L_X} + \lambda_{j,t}^{P_X} \right) \ln \left( \frac{X_{j,t}^{L_X}}{X_{j,t}^{F}} \right) + \ln(E_{st}^{FR})
\]

(17)

\[
\ln(C_{st}) = \sum_j \frac{1}{2} \left( \lambda_{j,t}^{L_X} + \lambda_{j,t}^{P_X} \right) \ln \left( \frac{W_{j,t}^{L_X}}{W_{j,t}^{F}} \right)
\]

\[
+ \sum_i \frac{1}{2} \left( \lambda_{i,t}^{L_X} + \lambda_{i,t}^{P_X} \right) \ln \left( \frac{Q_{i,t}^{L_X}}{Q_{i,t}^{F}} \right) - \ln(E_{st}^{FC})
\]

(18)

Since indexes, by definition, measure relative changes and are mathematically ratios, their logarithms are growth rates.8 Therefore, the first equation in (17) is the Fisher index decomposition of the

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7 Since we are assuming constant returns to scale and competitive profit maximization, technical change and TFP are the same.

8 In general, if \(x\) represents the index level or the relative change from \(s\) to \(t\), where \(t = s + 1\), then \(\ln(x)\) implies the growth equation, \(x = \exp[\ln(x)]\). By being the exponent of the base of natural logarithms, \(\ln(x)\) is, by definition, the logarithmic growth rate.
growth rate in revenues into the contributions from the growth rates of output prices, input quantities and TFP. The second equation in (18) is the Fisher index decomposition of the growth rate in costs into the contributions from the growth rates of input prices, output quantities and TFP. In these equations, the contributions of TFP growth are computed as residuals since everything else is known from data.

The detailed nominal growth decompositions in Equations 17 and 18 can be expressed equivalently and more compactly by,

\[
\ln \left( \frac{R_{st}}{P_{st}} \right) = \ln (X_{st}^F) + \ln (E_{st}^{FR}) \tag{19}
\]

\[
\ln \left( \frac{C_{st}}{W_{st}} \right) = \ln (Q_{st}^F) - \ln (E_{st}^{FR}) \tag{20}
\]

In this case, the bracketed ratio in the left-hand side of Equation 19 is the implicit output quantity index that exactly equals \( Q_{st}^F \), because of the Fisher factor reversal property. Hence, the logarithm of this ratio measures real output growth – i.e. growth in revenues less the growth in output prices. In turn, real output growth equals growth in inputs plus TFP growth.

By the factor reversal property, the bracketed ratio in the left-hand side of Equation 20 is the implicit input quantity index that exactly equals \( X_{st}^F \) and, therefore, its logarithm measures the growth of the real value of inputs – i.e. cost growth less the growth in input prices. In turn, growth of the real value of inputs equals growth in real output minus TFP growth.

Finally, the factor reversal property and the accounting identity (i.e. \( R_{st} = C_{st} \)) together imply that Equations 19 and 20 yield exactly the same TFP growth.

It may be noted that the real growth decompositions in Equations 19 and 20 describe the more common procedure. However, the results from Equations 19 and 20 are all obtainable from the growth decompositions in Equations 17 and 18 and, thus, the latter decompositions are more general.

**Törnqvist index decomposition of growth in revenues and costs**

The Törnqvist (denoted by superscript \( T \)) price and quantity indexes for outputs and inputs are,

\[
P_{st}^T = \prod_{i}^{M} \left( \frac{P_{i,t}}{P_{i,t-1}} \right)^{1/2(\gamma_{t,i} + \gamma_{t,i})} ; \quad Q_{st}^T = \prod_{i}^{M} \left( \frac{Q_{i,t}}{Q_{i,t-1}} \right)^{1/2(\gamma_{t,i} + \gamma_{t,i})} \tag{21}
\]

\[
W_{st}^T = \prod_{j}^{N} \left( \frac{w_{j,t}}{w_{j,t-1}} \right)^{1/2(\gamma_{t,j} + \gamma_{t,j})} ; \quad X_{st}^T = \prod_{j}^{N} \left( \frac{X_{j,t}}{X_{j,t-1}} \right)^{1/2(\gamma_{t,j} + \gamma_{t,j})} \tag{22}
\]

The revenue shares in Equation 21 are the same as those earlier defined in Equations 5 and 6. Also, the cost shares in Equation 22 are the same as those defined earlier in Equations 9 and 10.

Diewert and Morrison (1986) and Kohli (2003) showed that if the underlying aggregator function (i.e. revenue or cost function) is a flexible translog function and there is competitive profit-maximization, then exact Törnqvist TFP indexes exist given by,

\[
E_{st}^{TR} = \frac{R_{st}}{P_{st}^T X_{st}^T} = \frac{R_{st}}{P_{st}^T X_{st}^T} \tag{23}
\]

The first is Diewert and Morrison’s exact TFP index from the GDP (revenue) function where \( R_{st}/P_{st}^T \) is the implicit output quantity index. The second is Kohli’s exact TFP index from the national income (cost) function where \( R_{st}/Q_{st}^T \) is the implicit output price index. These yield a Törnqvist index representation of dual revenue and cost functions,

\[
R_{st} = P_{st}^T X_{st}^T E_{st}^{TR}; \quad C_{st} = W_{st}^T Q_{st}^T E_{st}^{TC} \tag{24}
\]

The results in Equation 24 are analogous to the Fisher index representation in Equation 14.

There is, however, a difference between Equations 14 and 24. Because the Fisher index satisfies factor reversal, the accounting identity ensures the equality of the Fisher TFP indexes for the revenue and cost sides, i.e. \( E_{st}^{FR} = E_{st}^{FC} \). In contrast, even with the accounting identity, the lack of factor reversal results in an approximation between the exact Törnqvist TFP indexes for the revenue and cost sides, i.e. \( E_{st}^{TR} \approx E_{st}^{TC} \).

For comparison with Equation 23, the direct Törnqvist TFP indexes are, by definition,

\[
E_{st}^{TR} = \frac{Q_{st}^T}{X_{st}^T} ; \quad E_{st}^{TC} = \frac{W_{st}^T}{P_{st}^T} \tag{25}
\]

The first is the primal revenue-side TFP and the second is the dual cost-side TFP. Because the Törnqvist index does not satisfy factor reversal,

\[
R_{st} \approx P_{st}^T Q_{st}^T ; \quad C_{st} \approx W_{st}^T X_{st}^T \tag{26}
\]

\(^9\) The Törnqvist TFP indexes in Equation 23 are exact algebraic derivations from a hypothesized flexible translog functional form for the theoretical productivity index. For details of the derivation, see Diewert and Morrison and Kohli mentioned earlier in this article.
The approximations in Equation 26 imply that the indexes in Equations 23 and 25 are related by,

\[
F^T_{st} \equiv \frac{R_{st}/P^T_{st}}{X^T_{st}} \approx E^T_{st} \equiv \frac{Q^T_{st}}{X^T_{st}}; \quad E^C_{st} \approx \frac{W^T_{st}/Q^T_{st}}{P^T_{st}} \approx E^T_{st} \equiv \frac{W^T_{st}}{P^T_{st}} \quad (27)
\]

That is, the lack of the factor reversal property implies in Equation 27 that the implicit output quantity, \(R_{st}/P^T_{st}\), and output price, \(R_{st}/Q^T_{st}\), indexes are not necessarily equal to their corresponding direct Törnqvist output quantity, \(Q^T_{st}\), and output price, \(P^T_{st}\), indexes. Moreover, as shown in Equation 28, the primal (revenue side) and dual (cost side) exact Törnqvist TFP indexes are not necessarily equal even though the accounting identity is satisfied. The same relationship holds between the primal and dual direct Törnqvist TFP indexes.

Finally, combining Equations 21, 22 and 24 yields,

\[
R_{st} = \prod_i^M \left( \frac{P_{i,s}}{P_{i,s}} \right)^{1/2(r_{i,s} + c_{i,s})} \prod_j^N \left( \frac{X_{j,s}}{X_{j,s}} \right)^{1/2(c_{j,s} + r_{j,s})} E^T_{st} \quad (29)
\]

\[
C_{st} = \prod_j^N \left( \frac{W_{j,s}}{W_{j,s}} \right)^{1/2(c_{j,s} + r_{j,s})} \prod_i^M \left( \frac{Q_{i,s}}{Q_{i,s}} \right)^{1/2(r_{i,s} + c_{i,s})} (E^T_{st})^{-1} \quad (30)
\]

By taking natural logarithms, Equations 29 and 30 yield,

\[
\ln(R_{st}) = \sum_i^M \frac{1}{2} \left( r_{i,s} + c_{i,s} \right) \ln \left( \frac{P_{i,s}}{P_{i,s}} \right) + \sum_j^N \frac{1}{2} \left( c_{j,s} + r_{j,s} \right) \ln \left( \frac{X_{j,s}}{X_{j,s}} \right) + \ln(E^T_{st}) \quad (31)
\]

\[
\ln(C_{st}) = \sum_j^N \frac{1}{2} \left( c_{j,s} + r_{j,s} \right) \ln \left( \frac{W_{j,s}}{W_{j,s}} \right) + \sum_i^M \frac{1}{2} \left( r_{i,s} + c_{i,s} \right) \ln \left( \frac{Q_{i,s}}{Q_{i,s}} \right) - \ln(E^T_{st}) \quad (32)
\]

In Equations 31 and 32, the TFP growth contributions are computed as residuals since everything else is known.

The above nominal growth decompositions can be expressed equivalently and more compactly by the real growth decompositions,

\[
\ln \left( \frac{R_{st}}{P^T_{st}} \right) = \ln(X^T_{st}) + \ln(E^T_{st}) \quad (33)
\]

\[
\ln \left( \frac{C_{st}}{W^T_{st}} \right) = \ln(Q^T_{st}) - \ln(E^T_{st}) \quad (34)
\]

Note that each equation is an exact decomposition. However, noting from Equation 26 that the Törnqvist does not satisfy factor reversal, Equations 33 and 34 do not yield exactly the same TFP growth even with the accounting identity (i.e. \( R_{st} = C_{st} \)). That is,

\[
\ln \left( \frac{R_{st}}{P^T_{st}} \right) \approx \ln(Q^T_{st}); \quad \ln \left( \frac{C_{st}}{W^T_{st}} \right) \approx \ln(R^T_{st}) \quad (35)
\]

As earlier remarked, real growth decompositions – by the Fisher Equations 19 and 20 or by the Törnqvist Equations 33 and 34 – are more commonplace. However, all the results from the above real growth decompositions are, respectively, obtainable from the nominal growth decompositions given by the Fisher Equations 17 and 18 or by the Törnqvist Equations 31 and 32 and, thus, the latter decompositions are more general.

**Closeness between the Fisher and Törnqvist growth contributions**

The decompositions of growth in revenues by the Fisher procedure in Equation 17 or by the Törnqvist procedure in Equation 31 add up to the same overall growth rate, \( \ln(R_{st}) \). Also, the Fisher and Törnqvist decompositions of growth in costs in Equations 18 and 32, respectively, yield the same overall growth rate, \( \ln(C_{st}) \). The component growth contributions may differ; however, between Equations 17 and 31 or between Equations 18 and 32. Regarding the contribution of TFP growth, the difference is due to the fact noted earlier that the Fisher index satisfies the factor reversal property but the Törnqvist does not satisfy this property. With respect to the contributions of the other components, the differences are due to differences in weights. To see these differences analytically, it is sufficient to compare the Fisher weights \( \lambda^L_{i,s} \) and \( \lambda^P_{i,s} \) in Equation 18 to the corresponding Törnqvist weights \( r_{i,s} \) and \( r_{i,s} \) in Equation 32.

Notice that Equations 18 and 32 are mathematically similar except for their weights. Their corresponding right-hand components approximate each other because in Equation 5,

\[
\lambda^L_{i,s} \approx r_{i,s} \text{ if } \left[ \frac{(q_{i,s}/q_{i,s}) - Q^L_{st}}{\ln(q_{i,s}/q_{i,s}) - \ln(Q^L_{st})} \right] \approx 1 \quad (36)
\]

Also, \( \lambda^P_{i,s} \approx r_{i,s} \) in Equation 6 if the term in brackets approximates 1. Therefore, the Fisher and Törnqvist
index procedures yield approximately equal growth contributions for the same component. As a key to this approximation, it is instructive as a preliminary step to establish that the arithmetic growth rates \([q_{i,s}/q_{i,s} - 1]\) and \((Q_{st}^L - 1)\) cannot be smaller than their corresponding logarithmic growth rates \([\ln(q_{i,s}/q_{i,s})]\) and \([\ln(Q_{st}^L)]\).

As noted by Balk (2004), Lorenzen (1990) showed that if \(q_{i,s}\) changes to \(q_{i,s}\), where these quantities are positive, the logarithmic mean \(L(q_{i,s}, q_{i,s})\) is bounded from below by the geometric mean and from above by the arithmetic mean. That is,

\[
(q_{i,s}, q_{i,s})^{1/2} \leq L(q_{i,s}, q_{i,s}) \leq \frac{(q_{i,s} + q_{i,s})}{2}; \quad (37)
\]

\[
L(q_{i,s}, q_{i,s}) = \frac{q_{i,s} - q_{i,s}}{\ln(q_{i,s}/q_{i,s})}; \quad L(q_{i,s}, q_{i,s}) = q_{i,s}. \quad (38)
\]

Now, substitute \(L(q_{i,s}, q_{i,s})\) in Equation 38 into Equation 37 and suppose that \(q_{i,s} > q_{i,s} > 0\). Then, divide the two sides of the left-hand inequality in Equation 37 by \(q_{i,s}\) to obtain,

\[
1 \leq \left(\frac{q_{i,s}}{q_{i,s}}\right)^{1/2} \leq \frac{[(q_{i,s}/q_{i,s}) - 1]}{\ln(q_{i,s}/q_{i,s})} \quad (39)
\]

Since \(q_{i,s} > q_{i,s} > 0\) in (39), \([\ln(q_{i,s}/q_{i,s})] > 0\). Using the latter to multiply all sides of Equation 39,

\[
0 \leq \ln\left(\frac{q_{i,s}}{q_{i,s}}\right) \leq \left[\left(\frac{q_{i,s}}{q_{i,s}}\right) - 1\right] \quad (40)
\]

Balk (2004) also noted that the logarithmic mean is bounded from below by \(\min(q_{i,s}, q_{i,s})\) and from above by \(\max(q_{i,s}, q_{i,s})\). Hence, suppose now that \(0 < q_{i,s} < q_{i,s}\). Therefore,

\[
q_{i,s} = \min(q_{i,s}, q_{i,s}) \leq \frac{(q_{i,s} - q_{i,s})}{\ln(q_{i,s}/q_{i,s})} \leq \max(q_{i,s}, q_{i,s}) = q_{i,s}. \quad (41)
\]

Dividing all sides of Equation 39 by \(q_{i,s}\) yields,

\[
0 \leq \left(\frac{q_{i,s}}{q_{i,s}}\right) \leq \frac{[(q_{i,s}/q_{i,s}) - 1]}{\ln(q_{i,s}/q_{i,s})} \leq 1 \quad (42)
\]

Since \(0 < q_{i,s} < q_{i,s}\) in (42), then \([\ln(q_{i,s}/q_{i,s})] < 0\). Hence, using this to multiply all sides of Equation 42,

\[
0 \geq \left[\left(\frac{q_{i,s}}{q_{i,s}}\right) - 1\right] \geq \ln\left(\frac{q_{i,s}}{q_{i,s}}\right) \quad (43)
\]

The aforementioned results show that arithmetic and logarithmic growth rates are either both positive in Equation 40 or both negative in Equation 43. In either case, however, arithmetic growth rates cannot be smaller than logarithmic growth rates. They are equal only if both are zero. These results apply to growth rates of components and to growth rates of aggregates or indexes as well. In general,

\[
\left(\frac{q_{i,s}}{q_{i,s}}\right) - 1 \geq \ln\left(\frac{q_{i,s}}{q_{i,s}}\right); \quad Q_{st}^L - 1 \geq \ln(Q_{st}^L) \quad (44)
\]

We now use Equation 44 to establish that the right-hand term in Equation 36 approximates 1. Consider that the first inequality in Equation 44 yields, by adding and subtracting terms,

\[
\left[\left(\frac{q_{i,s}}{q_{i,s}}\right) - 1\right] - (Q_{st}^L - 1) \geq \ln\left(\frac{q_{i,s}}{q_{i,s}}\right) - \ln(Q_{st}^L)
\]

\[
+ \left[\ln(Q_{st}^L) - [(Q_{st}^L - 1)]\right] \quad (45)
\]

Also, the second inequality in Equation 44 implies that the last term in Equation 45 is \([\ln(Q_{st}^L) - [(Q_{st}^L - 1)] \leq 0\), i.e. nonpositive. Therefore, by dropping this term and then grouping the remaining terms, Equation 45 yields either,

\[
\left(\frac{q_{i,s}}{q_{i,s}}\right) - Q_{st}^L \geq \ln\left(\frac{q_{i,s}}{q_{i,s}}\right)
\]

\[
- \ln(Q_{st}^L) = \ln\left[\left(\frac{q_{i,s}}{q_{i,s}}\right)/Q_{st}^L\right] \quad \text{or} \quad (46)
\]

\[
\left(\frac{q_{i,s}}{q_{i,s}}\right) - Q_{st}^L < \ln\left(\frac{q_{i,s}}{q_{i,s}}\right)
\]

\[
- \ln(Q_{st}^L) = \ln\left[\left(\frac{q_{i,s}}{q_{i,s}}\right)/Q_{st}^L\right] \quad (47)
\]

Note that both sides of each inequality in Equations 46 and 47 will always have the same signs.11 Therefore, the term in square brackets in Equation 36 can be equal to or greater than 1. It can also be less than 1 but it cannot be negative. That is, this term approximates 1 depending on the closeness between arithmetic and logarithmic growth rates from actual data. From our data on the three output categories and also three input categories in Tables 1 and 2, the average values during 1948–2001 of terms similar to Equation 36 are 0.986 for output prices, 1.022 for average values during 1948–2001 of terms similar to

10The inequalities in Equation 44 may be visualized as follows. The arithmetic percent growth of the Laspeyres quantity index, for example, is defined by the straight line, \(y = (Q_{st}^L - 1) \times 100\). The logarithmic percent growth is defined by the curved line, \(y^* = \ln(Q_{st}^L) \times 100\). The two equations are tangent at \(y = y^* = 0\) when \(Q_{st}^L = 1\), but the straight line lies above the curved line for all values of \(Q_{st}^L\) other than 1, i.e. \(y > y^*\) when \(Q_{st}^L \neq 1\).

11In Equations 46 and 47, if the difference between the two terms in the left-hand side is positive then in the right-hand side their ratio is greater than 1 so that the logarithm of this ratio is positive. Conversely, a negative difference in the left-hand side yields in the right-hand side a ratio less than 1 that has a negative logarithm.
The preceding analytical and empirical results imply, in principle, that the multiplicative Fisher index (as noted earlier in footnote 6) approximates the Törnqvist index. For the quantity indexes, for example, this means that \( \ln(QF_{st}) \approx \ln(QT_{st}) \). In general, these results underpin the finding in our revenue and cost growth decompositions (Tables 1 and 2) that the Fisher growth contributions approximate very closely the corresponding (i.e. the same component) Törnqvist growth contributions. In fact, the
corresponding contributions are equal up to 1/100 of a percentage point in most cases and, thus, appear virtually indistinguishable.

III. Application to the US Agricultural Sector, 1948–2001

We apply our decomposition of the growth in revenues and costs to the US agricultural sector over the period 1948 to 2001. The data used in the decomposition are very detailed and compiled using a production-theoretic approach described in Ball, et al. (1997). In this application, the data are aggregated into three categories of output—livestock and livestock products, crops and agricultural services – and three input categories – capital, labour and intermediate materials.

Table 1 reports the results of the decomposition of growth in revenues using both the Fisher and Törnqvist index number procedures. We see that the growth in revenues averaged 3.31% per year over the entire 1948–2001 period. TFP growth contributed an average 1.90% points to the growth in revenues and overshadowed the 1.43% point contribution of output prices. The overall level of input use actually declined, contributing a negative 0.02% points to growth in revenues.

Looking at revenue growth during subperiods, Table 1 shows that revenues fell since 1948 and started growing only in the late 1950s. Revenue growth continued during the 1960 decade and accelerated to a peak in the early 1970s and then decelerated until the last year of data in 2001. In each of the eight subperiods, the average annual contribution of TFP growth was positive and in most instances was the largest contribution to revenue growth. At the opposite, the revenue growth contribution of the labour input was negative in each subperiod. Moreover, the growth contributions of the capital input were mixed (i.e. positive in some subperiods and negative in others). Finally, the revenue growth contributions of materials were consistently positive except during 1980 to 90. As a whole, however, the net contributions of the three inputs to revenue growth were slightly negative over the full 1948 to 2001 period.

Since the data satisfy the accounting identity, the average increase in costs was also 3.31% per year. Table 2 gives the decomposition of the growth in costs based on the Fisher and Törnqvist index number procedures. The results show that the contribution of TFP growth was a negative 1.90% points, while growth in output quantities contributed 1.88% points and growth in input prices contributed 3.33% points to growth in costs.

As noted in Table 1, the revenue growth contribution of labour input was negative in each subperiod period, implying labour input declined. One reason could be higher relative prices, which is corroborated in Table 2 where the contribution of labour prices to growth in costs was positive in each subperiod and was the largest on average over the 1948 to 2001 period. Except in some subperiods, output quantities and input prices had positive contributions to growth in costs while TFP had negative contributions in all subperiods. Indeed, TFP contributed the most to slowing the rise in costs.

Note that subtracting the contributions of the growth in prices reported in Tables 1 and 2 from the growth in revenues or costs yields the rates of growth in the real values of output or inputs. Thus, after subtracting the growth contributions of output prices in Table 1, we find that the real value of output increased at an average 1.88% per year. The contributions of labour and capital to growth in real output were negative. While the average contribution of materials input to output growth was positive, this contribution was not sufficient to outweigh the negative contributions of labour and capital. Hence, the net contribution of all three inputs to output growth was slightly negative, implying that TFP growth was wholly responsible for output growth in the farm sector. Studies by Kendrick and Grossman (1980), Jorgenson et al. (1987) and Jorgenson and Gollop (1992) report similar findings.

Similarly, subtracting the contribution of input prices from the growth in costs gives the rate of growth in the real value of inputs. Performing these calculations confirms our earlier finding that the real value of inputs decreased at an average 0.02% per year over the 1948–2001 period. This comes from the fact that growth in output contributed 1.88% points per year while TFP growth contributed a negative 1.90% points to growth in the real value of inputs.

Finally, we note that the contributions of the components to growth in revenues or costs based on the Fisher index number procedure differ slightly, but only in very few instances from those based on the Törnqvist index. In fact, for the same component, the corresponding contributions are equal up to 1/100 of a percentage point in most cases and, thus, appear virtually indistinguishable. However, the growth contributions of all components from either index number procedure add up to exactly the same overall rate of growth in revenues or in costs.
IV. Conclusion

In this article, we proposed a decomposition of the growth in nominal revenues or costs that is analytically more general in yielding all the results from the common procedure focusing on real growth decomposition. We employed the superlative Fisher and Törnqvist indexes to provide an exact decomposition of the growth in revenues in US agriculture into the contributions of growth in output prices, input quantities and TFP. We assume that the underlying data were generated by constant returns to scale technology, so that factor payments just exhaust income. This allows a second decomposition of the growth in costs into the contributions of growth in output quantities, input prices and TFP.

We implemented the multiplicative or geometric mean form of the Fisher index that is mathematically similar to the Törnqvist index. This provided an alternative basis for their approximation properties by showing that these indexes differ in their weights mainly due to differences between logarithmic and arithmetic growth rates. Thus, they approximate each other closely depending on the closeness between logarithmic and arithmetic growth rates from actual data. Our findings on this basis complement Diewert’s well-known second-order differential approximation properties between the Fisher and Törnqvist indexes.

While we found slight differences in very few instances between the Fisher and Törnqvist growth contributions of the same component, these contributions are in most cases virtually indistinguishable. However, the growth contributions from either index number procedure add up to exactly the same overall rate of growth in revenues or in costs. These results imply that the mathematically simpler and computationally easier Törnqvist index framework is the more practicable procedure.

Our empirical results show that revenues grew an average 3.31% per year over the 1948 to 2001 period. Increases in output prices contributed 1.43% points per year to revenue growth, while growth in input quantities contributed a negative 0.02% points. The rather large increase in revenues relative to the increase in output prices reflects the increase in output made possible by the gains in TFP, which added 1.90 percentage points per year to growth in revenues.

On the cost side, growth in output quantities accounted for 1.88% points of the 3.31% rate of growth. Increases in input prices contributed 3.33% points per year, or slightly more than the overall increase in costs, while TFP growth contributed a negative 1.90% points to the rate of growth in total costs.

Our decomposition shows the contributions of growth in the components to growth in nominal revenues or costs. However, by subtracting the contribution of prices, we obtain the rates of growth in real values. Performing these calculations reveals that the real value of output increased. The contributions of labour and capital to growth in output were negative. The growth in materials input was positive, but the contribution of materials was not sufficient to outweigh the negative contributions of labour and capital, leaving responsibility for growth in output wholly to TFP growth. This result is consistent with earlier findings of the major sectoral productivity studies.

Finally, we note that revenue or cost measures income. Therefore, the proposed decomposition determines the contributions of growth in output prices, input quantities and TFP to income growth. This result has important policy implications because policies affecting output prices or input quantities may differ from those designed to boost productivity growth. Our application to US agriculture shows that TFP growth was the dominant source of growth in income over the 1948 to 2001 period. The contribution of inputs to growth in income was actually negative. We conclude that policies designed to boost productivity growth, such as increased investment in research, may be more effective than price or quantity instruments in enhancing growth in income.

References


Decomposition of growth in nominal revenues and costs


