THE CONSTRUCTION OF TAPER CURVES

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INTRODUCTION

Among the first requisites of a forester’s equipment in estimating standing timber are accurate and dependable volume tables, applicable to the stands being estimated and the utilization expected. Such tables show, in general, the average volume (in board feet, cubic feet, or other units of measure) of trees of different diameters and heights. It has not proved easy to construct dependable tables of this kind, owing to the difficulty of determining accurately the average volume of trees in the unusual size classes represented by very tall, slender trees and very short, large trees, for it is difficult to discover specimens enough of these classes to furnish the basis of reliable averages.

The first volume tables to be used were constructed simply by scaling a large number of trees of different sizes, averaging the scale of all trees in the same diameter and height class and then harmonizing the values for the different height classes as well as possible by graphic methods. This was the method described in 1906 by Graves (9, p. 158–163, 166, 167). It is slow, requires a large mass of data, and the graphic harmonization necessary to make values run smoothly often introduces grave inaccuracies.

In 1915, Barrows (1, 2) described a new method, which introduced taper curves, or curves showing the average form of trees, as a step in volume table construction. This was a great improvement over the earlier systems, and has been very generally used since that time. The method was accepted as undoubtedly sound and effective, the results were believed to be entirely satisfactory, and much time and labor were put in on the tedious series of curves necessary in this method. The present writer, however, in making a series of taper curves for lodgepole pine on a slender basis of trees, discovered that very palpable errors had come through the whole series of harmonizations and that the final curves were not dependable. This has led to an analysis of the underlying theory of Barrows’s series of recurvings, and to the conclusion that the latter are not as satisfactory as they would appear to be on the surface. It is no new discovery that errors exist in this method of handling taper curves. Barrows himself admits the fundamental error, that taper curves give not the average volume of the trees in any group, but the volume of the tree of average dimensions.

In finding average tree form, diameters are averaged; in the earlier method, volumes proportional to diameters squared were averaged. The error introduced depends upon the range of values in a given class, but, as pointed out by Barrows, it never becomes serious. Bruce (4) implies more grave dangers by demonstrating the presence of errors in certain volume tables undoubtedly built up according to Barrows’s method. He shows that in these tables the values vary erratically from the volumes of the frustums of cones having the same top diameters and the same breast-high diameters as the trees; a divergence that can not possibly be true if the values pretend to represent true means. The cause of this variation, however, was never determined by Bruce, which leaves an uncertainty as to just what is wrong with taper curves, and whether the difficulty is remediable or not.

NATURE OF UNHARMONIZED TAPER CURVES

In analyzing Barrows’s method and the errors inherent in it, consideration must first be given to the basic data from which the taper curves are drawn. These consist of individual tree records giving diameter, usually by regular intervals (often 8 or 16 feet) from the stump to the top of the tree. Height is given to the nearest tenth of a foot; diameter to the nearest tenth of an inch. The size classes which are used as the basis of all computations, and which appear in the final volume tables were not dependable. This has led to an analysis of the underlying theory of Barrows’s series of recurvings, and to the conclusion that the latter are not as satisfactory as they would appear to be on the surface. It is no new discovery that errors exist in this method of handling taper curves. Barrows himself admits the fundamental error, that taper curves give not the average volume of the trees in any group, but the volume of the tree of average dimensions.

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Received for publication June 30, 1924; issued June, 1925.

1 Reference is made by number (italic) to “Literature cited,” p. 624.
are usually based on 1-inch diameter intervals and 10-foot height intervals. For example, a tree 14.7 inches in diameter at breast height and 82.2 feet tall, belongs in the 15 inch–80 foot class. Trees are usually well distributed as to diameters within a given inch and height class so that the average of all trees falling in such given class is very close to the assigned even diameter. Of course, for each height class there is a mean diameter from which the diameter values fall away on each side roughly in accord with the laws of chance. This curve is long and flat, however, for diameters in any height class, and the distribution of diameters through any single inch class is virtually even, although, of course, theoretically not quite so in any but the model class. Thus, in the 12-inch d. b. h. class, individual trees will range from 11.6 to 12.5 inches d. b. h. Unless there are very few trees indeed in the class, the distribution will always fall very close to 12.0 inches. Only occasionally in the highest and lowest diameter classes will this rule fail, as it may indeed by chance in any class represented by a very few trees.

Conditions are not the same, however, in regard to the height. Trees of the same inch class may be tall, short, or medium, and a very wide variation of height values is possible, ranging from two-thirds to one-half of the total height of the trees, in lodgepole pine at least. This variation is very nearly "normal," that is, the heights vary from the central or modal value according to the laws of chance. In case, therefore, the entire sample of data relating to a single inch class is subdivided into five smaller classes on the basis of height, the central class will have evenly balanced values and the actual average will generally represent the true mean of the class. The next class adjacent both above and below will have its values piled up asymmetrically, with a preponderance of values nearer the average height for the inch class. For example, suppose there are five height classes, 100, 90, 80, 70, 60 feet, as shown in Figure 1. The average height of all trees in the central class, 80 feet, will be 80 feet. In the 90-foot class there will tend to be more 86-foot trees than 94, and the average height of the 90-foot class will be lower, perhaps about 88 feet. Conversely, in the 70-foot class values will tend to run high. In the outermost classes this effect is still more pronounced and the average 100-foot tree may be only 96 feet tall, while the average 60-foot tree may be 64. If the tables were to be used only in the region where prepared, this would be of little moment, for in the timber estimate 100-foot trees would again average 96 feet tall and the volume of 96-foot trees would be the correct volume to use. In a region where sites were better, however, trees in the 100-foot class might be 100 feet tall and volumes based on 96-foot trees would be low.

It must be the aim of taper curves and volume tables to give the true middle values of the class represented, or else they will be meaningless. Therefore, the values of the 15 inch–100 foot tree class must be exactly right for trees 15 inches in diameter, and 100 feet tall, and not perhaps for a tree 14.8 inches d. b. h. and 96.7 feet in height. One qualification of any system of volume-table construction must be its ability to take these values that fall a little from the middle of the class they represent and bring them into their proper places.

When diameters taken at intervals up the tree are averaged in each diameter and height class, as is the first step in the preparation of taper curves, there is some question as to the value these averages have and how nearly they truly represent the midtree of the class. This apparently simple operation is not without its complexities.

Sometimes in getting average diameters well toward the top it will be found that certain trees "drop out." For example; if the diameter 78 feet from the base in the 80-foot class is being sought, some trees will "drop out," being less than 78 feet tall and yet over 75 feet tall, and therefore still in the 80-foot class. In averaging they usually are given a value of zero, but it is easily seen that they have a potentially minus value. So, where some of the trees have "dropped out" average diameters are too high. It is well to disregard measurements of diameters at heights where all the trees in the class are not represented, as they lead to error rather than accuracy.

Aside from these mathematical points is the natural fact that the greatest variability in trees is toward the tops. In any given inch and height class the diameter of the trees will vary little more than an inch between maximum and minimum diameters 8 feet above the ground, and perhaps about 2 inches at 16 feet up; while 72 feet up in an 80-foot height class the diameters may range over 5 or 6 inches in variance even in smooth normal trees. Almost every mass of field data contains abnormal trees measured too near swell-
ings, large limbs, forks, etc., which all contribute to wide variability upward. These factors in turn lead to mean diameters containing large probable errors, so much so that when a series of them in consecutive size classes is compared, a great lack of harmony is always found, the amount depending chiefly upon the number of tree measurements used as a basis.

HARMONIZING THE CURVES—BARROWS'S METHOD

Having examined the shortcomings of the original data, the next step is to see how the method of compilation gets around them.

The conventional method of Barrows consists of several series of curves, of three primary forms shown in Figure 2 in a somewhat simplified form, conical trees and straight lines being used to show more clearly certain points.

The first series of curves, exemplified in Figure 2, A, is drawn through points representing the average diameters at different heights, as compiled from the original tree measurement sheets. All trees of various diameter classes are thrown into one chart, a separate chart being made for each height class. Drawing curves through the plotted points serves to iron out minor irregularities of form and makes the average tree a smooth curve. These curves do not necessarily pass through the exact middle value of each d. b. h. class, but may vary from the exact even inch; although, as already pointed out, this is unusual, except where only a small number of trees are measured. Average height may also, and very frequently does, fall on one side or another of the exact middle point of the height class as shown in Figure 1. Nowhere does Barrows intimate that this occurs, although he particularly points out the more infrequent case of the average tree failing to fall on the even inch of the proper d. b. h. class.

One fact that helps to prevent the errors in the upper parts of taper curves from becoming extremely serious is the stability of the end point derived from averaging the total heights of the trees.

At the same time the possibility of errors in the higher parts of all curves presents a considerable problem. Such errors all contribute to irregularities and peculiarities of shape in individual curves which can only be properly straightened out by an efficient method of harmonization.

Figure 2, A, shows the first series of curves simply by the use of straight lines. Accepting Barrows' statement, they are shown failing to pass through the middle value in each inch class (the even inch) at breast height, but at present for simplicity's sake are assumed to converge exactly at the 70-foot point.

The second series of curves is shown in Figure 2, B. In this series of curves the d. b. h. of the tree is made the...
absissa, and the diameter at other points the ordinate. As a rule, values are read at 10-foot intervals up the tree from the curves of series A, and a separate curve is drawn showing the relation of d. b. h. to diameter, 10, 20, 30, and 40 feet above ground level. This series of curves so rounds out the values in Figure 2, A, that, in effect, it takes the lines as shown in Figure 2, A, and turning them upon the point of convergence as a pivot spaces them systematically, placing them upon the correct d. b. h. If the lines in Figure 2, B, are straight, the corrected spacing of the lines in Figure 2, A, will be even; if they are curves, the spacing will be uneven, but the changes will be regular and even (as will occur in cases where form changes with diameter). In practice, lines in Figure 2, B, are nearly straight. Their slope depends upon the distance between the lines in Figure 2, A, at various levels, so there is the greatest slant at the base of the tree where the lines are farthest apart and the least toward the tip.

The spacing between the lines in Figure 2, B, depends upon the slope of the lines in Figure 2, A (taper of the trees). In conical trees they are naturally evenly spaced, as shown in the figure, but in full-boled trees they get farther and farther apart toward the top of the tree, as in such trees rate of taper increases upward. In practical work, the points determining these lines usually fall considerably of falling into good lines, so that in drawing an average curve errors are very likely to be made in giving end values with a poor basis more weight than they deserve, consequently tilting the lines. Marked tilting in any single line will be noticed on inspection of the finished graph, however, unless all the lines are tilted in the same direction (which unfortunately is quite possible). The effect is equivalent to making the lines in Figure 2, A, evenly spaced, but either too far apart or not far enough. The spacing may be different in different height classes, as these are curved entirely independently.
The third system of curves is aimed to harmonize values in the same d. b. h. class, but in different height classes. There are never many height classes in a single d. b. h. class, for the variation in the height of trees of a given diameter never amounts to many ten-foot intervals (the usual classification). The curves are, therefore, short and more open to error in direction than long curves. Their form is illustrated in Figure 2, C, and their function is similar to curves of the second series as shown in Figure 2, B, except that d. b. h. is constant instead of the height of the tree, and separate charts are drawn for each d. b. h. class. Their shape is different, however. In Figure 2, A, it is clear that the normal distribution of the lines is fairly evenly spaced across the sheet, at any given height, hence their relation is expressed by very nearly a straight line. In Figure 2, D, the spacing is even, but up and down rather than horizontally across the page. Curves as shown in Figure 2, C, however, if properly drawn, ought to have the effect of arranging the lines harmonically across the page, as is desired.

If these curves are not properly drawn they lead to a variety of errors. If the curve is the wrong shape (provided the end points are right) it means that in trees of medium height the diameters are incorrect. Curves which touch the basal line fortunately have one point fixed, for diameter at the top of a tree must always be zero. The curves representing points 10 and 20 feet above the ground approximate straight lines and divergence is slight, but higher curves are difficult to draw. In the first place, the values high in the tree are erratic, for reasons that will be shown later, and the curvings in series B do not remedy matters very much. Secondly, these points are incapable of much correction by the curves, for the change of direction is rapid at that point, divergence is considerable, and the curves are widely spaced, all of which renders the proper placing and shaping of the curve very difficult.

In practice it is found that the curves are most naturally drawn to pass through the point plotted to represent diameter 10 feet from the top, while actually such points often need severe correction. An error in diameter 10 feet below the top may not exist in one diameter class alone, as such a possibility has already been ironed out in the curves of Figure 2, B. Consequently, curves of Figure 2, C, pass it on, although theoretically they ought to remove it. For instance, in Figure 2, C, let us suppose the diameters of all the trees in the 50-foot class at the 40-foot point are high because of the misplacement of the 40-foot line in the previous curving of the type shown in Figure 2, B (for the regularity of the spacing of these lines is difficult to judge on a large scale). Then in Figure 2, C, the point A will fall high at the 40-foot point. In 40-foot and 60-foot tree classes (points B and C) the values we will suppose to be correct. Nevertheless, because of the rapid curvature, lack of parallelism and width of spacing in that part of the graphs, it is hard to make a choice between the solid line or the dotted line as shown in the figure, one confirming an erroneous value and changing a correct one, the other changing the wrong value to the right one. Curves of the form shown in Figure 2, C, serve to even up the erratic values, however, especially in tall trees and in the lower part of the bole, and to make changes run smoothly from height class to height class within the same inch class.

As each inch class is adjusted separately, there is no assurance that the same relative adjustment will be made in the neighboring inch classes. Accordingly, values are read back again to make curves of the type shown in Figure 2, B, where the inch classes are again put in harmonious relations with each other, with some possible disarrangement of ideal relations existing between the different height classes. These curves are finally transformed back again to curves of the type shown in Figure 2, A, the finished taper curve.

Having briefly outlined the method and noted some of the difficulties encountered, its practical effectiveness must be considered. The first curving makes the form of the trees in each diameter and height class independently smooth, although their forms may differ. This is good. Next, the lines are spaced evenly and are made to run through the right d. b. h. through the process shown in Figure 2, B. But although spaced evenly, what is the assurance that the spaces are the right distance apart? If wrong in a single height class, curves of series 3 (fig. 2, C) will tend to iron out the trouble when the erroneous value is placed in comparison with correct values in adjacent classes. But just here is where one error comes in.

In practice, the first curves fail to converge as they are shown doing in
Figure 2, A, upon the exact height class. Seventy feet may be tall for 9-inch trees and the average height of the 9 inch–70 foot class may be only 67 feet. Likewise, 70 feet may be short for a 16-inch tree and the average height may be 72 feet. So, ultimately, in a given diameter class, short trees are forcibly stretched out to the mean of their height class and tall trees are pulled down to the mean of the class, as already brought out and shown in Figure 1 and Figure 3 A, an idealized form of Figure 2 D, which leads to a distortion of form.

In practice, in drawing the first curves of the form shown in Figure 2 A, an irregular jumble and crisscrossing of lines in the upper part of the graph always occurs. Theoretically the result should look something like Figure 3 B, but owing to miscellaneous errors and differences in form toward the tree tips, instead of a mere failure of the lines to converge on the mean height value of the class as shown, there is a tremendous tangle of crossing lines, for there may easily be 20 inch classes in a single height class. When this is straightened out graphically and each curve forced to end at the mean value of the height class, it results in a much wider spread of values toward the top than there should be. The second curving thus straightens out the crisscrossing, but leaves the total spread of the lines about the same; the spacing is made even, but abnormally wide, so that small trees are forced to take on abnormally slight taper toward the top and big trees are made to show a great taper. The effect extends to a certain degree all the way down to breast height and tends to make the diameters of small trees (short for their height class) run small, and large trees (tall for their height class) run high. All this tends to make trees in extreme size classes show a form they do not actually possess.

In the next curving (fig. 2, C) values of trees in the same diameter class but in different height classes are combined. On every height line above the ground, values will run low in the short trees and high in tall, which merely shifts the tilt of the curves, a matter which is absolutely invisible when all are tilted equally or even harmonically. So this error passes on through to the end. Thus incorrect values caused by wrong spacing for any reason whatsoever of lines in Figure 2 A, tend to persist, as actually occurred in one instance of lodgepole pine taper curves.

The space between the lines is greater in the 70-foot class than in any other in the finished taper curve. This could be avoided to a certain degree by harmonizing the curves on a different basis to eliminate the inefficient third curve. Several alternative methods investigated by the writer, however, while more sound theoretically on account of the use of almost straight line curves throughout, proved practically as unsatisfactory in actual use and involved a great deal more labor than the system of harmonization outlined by Barrows.

The only conclusion possible from this study of Barrows's taper curve construction is that by it are readily obtained harmonized values that change...
smoothly from class to class. The system, however, can not claim a great degree of accuracy. In fact, it tends to sacrifice accuracy to harmony. To work best, not only a large basis of individual measurements is needed, but they must be well distributed through many diameter and height classes so that the resulting curves may be long and their trend be clearly shown, minimizing the ever present danger of tilting the curves into incorrect positions.

The object of this criticism of the method of preparing taper curves is not to throw doubt upon the accuracy of the volume tables prepared in accordance with this system (which have proven fairly satisfactory) but to show that it is not by any means perfect and that equal accuracy may be obtained by simpler means.

**FRUSTUM FORM FACTORS—BRUCE’S METHOD**

One obvious way of getting around the difficulties of Barrows’s method is to set a standard form of tree and compare all others with it. This is the essence of Bruce’s frustum form factor idea. The difference between the volume of a tree and the volume of a frustum of a cone having the same top diameter and the same d. b. h. ought to vary slowly and consistently with the changes of diameter, height, and form. Sudden changes are inconceivable where average trees are concerned. The frustum form factor method of volume table construction has been employed by Bruce and has proven most excellent indeed. It is easy, quick, and, as demonstrated by Bruce and others, is surprisingly accurate. It has the disadvantage, however, of what may be termed inflexibility. The final results come in one step from the basic data, and if there are any other results desired they must be worked up anew from the original measurements.

If a volume table is desired running to a 6-inch top limit, the volume of each tree to that limit must be figured and compared with the corresponding frustum. The frustum form factors must be averaged, and the computation for volume be made from them. If now a new table is desired showing volume to a 7-inch limit, the whole process must be repeated. With taper curves, the preparation of new tables is a matter of minutes. Furthermore, the taper curve system is the only one applicable to linear products, ties, props, etc.

Bruce’s frustum form factor is an empirical sort of figure bearing little relation to other usually accepted tree form constants, as form factor, form quotient, form exponent. One reason is that diameter breast high outside of bark is made one of the points through which the frustum surface passes, so that the frustum form factor will vary with bark thickness, if all other factors as height, diameter, and form remain constant. Thus it is not a measure of form in the sense that some other factors are.

The frustum form factor values also will vary with the top cutting limit used, because the lower part of a tree has more conical taper and the frustum of a tree cut to a 10-inch limit will fit much more closely to the frustum of a cone than when the top limit is 6 inches, well in the top of the tree. It is possible for two tree frustums of different form and equal top and basal diameters to have the same volumes, and hence the same frustum form factors. This factor is thus an expression of volume relations rather than form relations. It is a very useful empirical figure, and the basic idea is sound. Nevertheless it fails to fill the place occupied by the taper curves. It is still worth while to discover, if possible, improved methods of curve construction.

**MATHEMATICAL EXPRESSION OF TREE FORM**

A simple mathematical expression of form, and a generalized equation of tree curves has been sought by European foresters for many years, but none has proven entirely satisfactory. There have been no attempts of this kind in America except the recent modification of the Höjer formula, worked out by Behre for western yellow pine in Idaho, which is of too recent introduction to have proved its general usefulness as yet. The discovery of a simple general curve equation of this kind would go far in solving tree mensuration problems. In the search for such an equation, however, certain facts have been discovered which pave the way for a much simpler method of expressing tree form through taper curves.

**PROPOSED SUBORDINATE FORM QUOTIENT METHOD**

Tor Jonson, according to Claughton-Wallin, has proved that the taper of the trees of the same form class (form quotient) is independent of height with certain north European conifers. (The
fact that the taper is the same for trees of all diameters within the same height and form class had already been proved by Maass, Schiffl, and others.) Whether Jonson's dictum is strictly true or not is of little moment. It is sufficient to say that by application of this hypothesis he has developed a wonderfully accurate system of timber estimating in Sweden. If accurate enough for European conditions, it should be ample for our needs.

Accepting Jonson's statement, therefore, that taper of trees of the same form quotient is the same from the top to bottom, we have only to gather all trees of the same form quotient together and determine the diameter at regular intervals up the stem in each class. The base of the tree is considered to be at breast height, and the middle diameter is at a point halfway between breast height and the tip of the tree.

Given the usual series of taper measurements, as collected in the field, the method of curve construction is not difficult. The different steps are illustrated by actual cases taken from the preparation of curves for Douglas fir in southwestern Idaho, which have been built by this method.

These taper curves were built from a slender basis of trees as an experiment to see how the new method would handle a difficult case. The basis was 1,123 trees scattered from the 12 inch–60 foot class to the 48 inch–150 foot class, a total of 136

Since trees of various heights are thrown together, the points of measurement must be at equal fractions of the tree height; and since various diameters are thrown together, the diameter at any point must be expressed in terms of diameter at the base of the tree (or some other fixed point). These figures, expressing diameter at various fractions of total tree height in terms of basal diameter, are very similar to the form quotient, which is this figure in the special case when the upper diameter is taken at half the height. Diameters at other points expressed in terms of basal diameter may with propriety be designated as subordinate form quotients. It must be remembered that here we are dealing all along with absolute form quotients in which the different individual diameter-height classes being involved, or an average of only about 8 trees to a class. The largest single-size group—the 16 inch–90 foot class—contained only 35 trees; and 35 classes were represented only by single trees. It is obvious that a difficult test was imposed. It must be borne in mind also that the exact figures and form relationships found are not to be considered as fundamental and must not be used as the foundation of any generalized concepts of tree form—even of Douglas fir, for there are, as a matter of fact, a number of fundamental details that appear unsound. This work is to be considered only for what it is—an attempt to use a new method in building up a practical set of taper curves.

![Crude taper curve for 16 inch-100 foot class Douglas fir. Numbers at points indicate number of measurements averaged](image-url)
The sheets carrying the original data gathered in the field are sorted in the usual way into diameter and height classes. The diameters are then averaged and taper curves are drawn for each height class, based on diameters inside of the bark (fig. 4). Inside diameters are almost never taken at breast height. Accordingly, the curve is simply brought down from the first measurement to the base line of the graph by eye. The stump diameters should not be plotted, and no stump flare or basal swelling be allowed for, as it will tend to cause later difficulties. It is indeed much better to leave out the actual diameter inside of the bark at breast height, even if the figure is available, as on large trees it may be affected by stump swelling and introduce error into the curves at this time. Of neighboring size classes, is open to a certain amount of scientific criticism. In Europe, form is generally held to vary with the density of the stand and to vary almost as widely in a single size class as in a whole stand. It therefore becomes most proper to assign a single form quotient to the whole stand on the basis of density. Possibly we would do well enough by simply throwing all size classes together and taking a single average form quotient. In the work done by the writer, however, there is a fairly regular trend of form-quotient values according to size, the quotients falling as diameter increases (in Douglas fir) and increasing with height. Possibly this is due in part to a rough correlation of size with density in many of our unmanaged forests; but, whatever the cause may be, the result is quite apparent, and consequently the form quotients of the various size classes have been harmonized by curves.

The values for each height class may first be conveniently smoothed separately, as the form quotient tends to change very slowly within the same height class with the species studied by the author. (See fig. 5 for the 100-foot class, South Idaho Douglas fir.) There are also many diameters in each height class, which gives much longer curves than would be obtained if an attempt were made to draw curves for each diameter class separately. All of these curves should then be plotted upon a single sheet so that the relations between height classes may be judged, and any necessary adjustments made, for it is obvious that values should not run

![Fig. 5.—Hundred-foot class Douglas fir, showing method of smoothing irregular form-quotient values. Numbers at points indicate number of trees as basis.](image-url)
irregularly from height class to height class. In all the work done by the author little adjustment of this kind has been necessary.

From these final curves the true absolute form quotient for each size class may be read. The next step is to select a number of the original taper curves which have a good basis in number of trees and yet show a wide variation in form quotients. After selecting 20 or 30 of these, the distance between breast height and the tip of the tree should be divided into 10 equal parts (fig. 6). The diameter of each one of these points up the tree is then read off and put down on paper. After this is done, each of these diameter values is divided by the d. b. h. inside bark of the tree from which they were taken, this being done very rapidly and with ample accuracy by slide rule. In this way a series of subordinate form quotients are obtained, showing the diameter at 10 points equidistant up the tree in percentage of diameter breast high.

In order to even off these values a series of curves should be made, with the form quotient as the abscissa and the subordinate percentages or form quotients the ordinate. All the values which have been computed should be plotted on a single graph according to the appropriate form quotient of the tree, as shown in Figure 7. In the example of Douglas fir, here used, they fall into very good lines and require little study in order to draw the proper curves. The curve at 0.5 of the way up the tree must be a straight line because that is the form quotient line. The lines representing the diameter percentage at 40, 30, 20, and 10 per cent of the height of the tree also apparently fall in straight lines, which when extended meet at ordinate value 1.0 and abscissa value 1.0. At 60, 70, 80, and 90 per cent of the tree height they are flat smooth curves.

After drawing these curves it is possible to read off the proper subordinate form quotients for any diameter and height class after reading the proper form quotient from the curves already prepared (as in fig. 5 for the 100-foot class). It is a simple thing then to prepare a taper table if the proper base diameter is known for each d. b. h. class of trees. This can easily be ascertained by plotting the d. b. h. inside the bark from the curves first drawn against the d. b. h. outside the bark for each diameter class (fig. 8). Variations can easily be smoothed out by simple curving.

Now, having the correct diameter inside the bark, the correct form quotient, and the correct subordinate form quotients all the way up the tree, it is a simple matter to draw smooth and correct taper curves by applying the
appropriate percentage to the proper diameter inside the bark (fig. 9). To anybody familiar with curves of similar form made by the conventional system, a striking difference will be noted in the upper part of the tree, where all the curves converge into one. A wide spread is the rule in the curves made by the usual system. In effect, this harmonize the whole curves themselves by the graphic methods employed by Barrows. The results are, furthermore, very much more dependable. This method also obviates any necessity of working out mathematically the equation of these curves.

It is interesting to compare the taper found by the method here used for

![Figure 7](image-url)

**Fig. 7.** Subordinate form quotients at ten equal intervals from the base of the tree to the top for Douglas fir trees having form quotients from 0.64 to 0.80

method gives the form or taper curve of all trees having the same form quotient, and each size class of trees is assigned its tree form simply by working the form quotient into a regular, orderly sequence. It is obviously much easier to bring these simple figures into order than to arrange, rearrange, and Douglas fir and the taper found by Behre for western yellow pine, using his modification of Höjer's equation. The difference is very slight. Taking Behre's form class 70 and comparing it with the values for form quotient .700 in Table I, which is based on the most complete data, the following results are obtained.
Fig. 8.—Relation of d. b. h. outside of bark to d. b. h. inside of bark

Fig. 9.—Douglas fir 60-foot class taper curves (12 to 16 inches d. b. h. inside bark)
The Construction of Taper Curves

TABLE I.—Comparison of taper for Behr’s western yellow pine form class with the taper for Douglas fir, form quotient .700

<table>
<thead>
<tr>
<th>Percentage of d. b. h. (inside bark)</th>
<th>Percentage of d. b. h. (inside bark)</th>
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<tbody>
<tr>
<td>Western yellow pine *</td>
<td>Douglas fir b</td>
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<tr>
<td>Tip to breast height</td>
<td>Tip to breast height</td>
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<tr>
<td>Per centage of length from</td>
<td>Per centage of length from</td>
</tr>
<tr>
<td>90</td>
<td>96.0</td>
</tr>
<tr>
<td>80</td>
<td>90.6</td>
</tr>
<tr>
<td>70</td>
<td>85.0</td>
</tr>
<tr>
<td>60</td>
<td>77.7</td>
</tr>
<tr>
<td>50</td>
<td>70.0</td>
</tr>
</tbody>
</table>

* Behre.  b Baker.

TAPER CURVES BASED ON THE PARABOLOID FORMULA

The system of curve construction just described was evolved because a still simpler system failed to work on Douglas fir, although it gave excellent results with both lodgepole pine and aspen. This involves the assumption that the taper curve follows the generalized formula for a paraboloid: \( Y = px^n \), or as it is more often written in forestry literature, \( Y^2 = px^r \). In this equation \( Y \) = diameter, \( x \) = distance from the top of the tree, and \( p \) and \( r \) are constants. This formula gives a series of parabolic curves varying in shape with the value of \( r \), and in steepness with the values of \( p \).

The first question that naturally arises is, how much risk of introducing error we take by the use of this hypothesis. Judging from Graves’s “Men-

suration,” (9) it is used widely in all Europe, and exists in an implied or approximate form in many formulas for determining the volume of trees. Table II is presented to show the comparison of the theoretical form with the actual average form in the case of the 60-foot height class in lodgepole pine. The curve \( Y^2 = px^r \) was based upon d. b. h. and d. h/2 in this work. Column A gives the actual average values (unharmonized by curves) in each class, column B the theoretical values. It is obvious that neither is an absolute value; there is a probable error in each. This has been figured for the actual values and it runs very close to ±0.1 inch, except in the 14-inch class, where it is nearer ±0.2 inch. The probable errors of the computed values in columns B are of approximately equal magnitude.

It is very evident that the error in assuming the trees to be paraboloids lies well within the probable error of ±0.1. Hence it is obvious that the form of lodgepole pine is virtually a paraboloid, at least in the 60-foot height class. Tests similar to that shown in

TABLE II.—Comparison of theoretical form with the actual average form for 60-foot height class lodgepole pine

<table>
<thead>
<tr>
<th>D.b.h. (ins.)</th>
<th>8</th>
<th>16</th>
<th>24</th>
<th>32</th>
<th>40</th>
<th>48</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>10</td>
<td>9.6</td>
<td>9.5</td>
<td>8.9</td>
<td>8.8</td>
<td>8.2</td>
<td>8.0</td>
</tr>
<tr>
<td>11</td>
<td>10.5</td>
<td>10.5</td>
<td>9.6</td>
<td>9.7</td>
<td>8.9</td>
<td>8.8</td>
</tr>
<tr>
<td>12</td>
<td>11.4</td>
<td>11.4</td>
<td>10.5</td>
<td>10.4</td>
<td>9.3</td>
<td>9.3</td>
</tr>
<tr>
<td>13</td>
<td>12.2</td>
<td>12.2</td>
<td>11.2</td>
<td>11.1</td>
<td>9.9</td>
<td>9.9</td>
</tr>
<tr>
<td>14</td>
<td>13.0</td>
<td>12.2</td>
<td>11.7</td>
<td>10.4</td>
<td>10.3</td>
<td>8.8</td>
</tr>
</tbody>
</table>

A = Actual average values, unharmonized by curves.  B = Theoretical values.

The above table was made in a variety of diameter and height classes where a large number of tree measurements were available and the results were equally conclusive in every case. Accordingly, it was assumed that the
same fact applied to all classes, and that any failures to agree were due to insufficient numbers of trees as a basis of computation. Curves were constructed upon the hypothesis. It may be noted in passing that these curves conform very closely to lodgepole pine taper curves already published (11).

In order to take advantage of this short-cut method it is only necessary to discover whether the taper curves in the size classes having the best basis in trees conform to the theoretical paraboloid curve. If they do so conform, the short method will suffice. If they do not conform, the method described before can be used. The methods of compilation start the same in either case. The original sheets are thrown into diameter and height classes as usual, the measurements inside bark are averaged, and separate rough taper curves are constructed for each diameter and height class. The form quotients are then computed, and the values smoothed by curves, as in Figure 6. Then, choosing several size classes where the basis in trees is heavy, the general correspondence between the theoretical curve and the actual curve should be noted. If it is close, the general similarity of all size classes to the paraboloid can be assumed. Then if $FQ = \text{form quotient}$, $r = \frac{-\log FQ}{0.15}$

$r$ being the exponent in the parabolic formula $Y^2 = px^r$.

The value of $r$ always falls near 1. Then if $x$ equals the distance from breast height to the top of the tree and $Y = \text{diameter breast high inside bark}$, $Y^2 = px^r$, from which $p$ can be ascertained. Then by use of this same equation, by introducing different values for $x$, the diameter at various heights can be readily ascertained.

The process sounds complicated, but can be done rapidly when once mastered, especially with the aid of an alignment chart. Such a chart is illustrated in Figure 10 (6). It may be mounted on a drawing board and arranged with a strip of celluloid bearing a straight line pivoted with a thumb tack moving on the $FQ - R$ line, and a thread fastened to a pin moving on the $P$ line (fig. 10).

To solve the equation by this chart, take any diameter and height class—say the 60-foot height class—look up the form quotient for that class and pivot a line on the celluloid strip at the appropriate value. Make this line pass through the height of the tree minus 4½ feet (the distance from the tip of the tree to the "base" at breast height) on the $H$ line. Then take the pin and thread and stick the pin in the value of diameter breast high inside bark for the particular class you are dealing with. Make the thread intersect the line on the celluloid strip on the unlettered and ungraduated line and stretch on to intersect a certain value of $p$, which should be marked. Then reverse the process. Stick the pin at the marked value of $p$. To find the diameter of the given tree 10 feet from the tip, pivot the celluloid line till it cuts 10 on the line $H$. Then stretch the thread from $p$ across the intersection of the celluloid line and the plain line, until it intersects a value on line $D$, which is the diameter 10 feet from the tip of the tree. The method is rapid.

The advantage of this method, as well as the one first outlined, over the older method lies in their ability to iron out errors (especially those due to failure of the data in any diameter and height class), and to average up to the middle of the class, while the results are expressed in the same useful taper curves. Their advantage over the system of frustum form factors lies in the fact that they have the same ability to get accuracy from scant data and are only slightly more laborious, while the results are expressed in taper curves instead of board feet. The wide usefulness of taper curves in all kinds of volume computation, yield, etc., is too well known to need enlarging upon.

The last method is also useful, for if trees are proven paraboloids, a very simple relation exists between form quotients and the regular conventional form factors, by the use of which total cubic contents can be very easily computed.

**CONCLUSION**

The whole subject of tree form needs deeper study so that the fundamental laws may be learned, which will lead to further simplification of methods in volume table construction. This study represents only the development of a simple but nevertheless empirical method based upon what is known at present of tree form, i.e., that trees of a given species having the same form quotients have the same form from top to bottom (excluding basal flare), and the very safe hypothesis that form (as expressed by the form quotient) varies regularly with changes in diameter and height.
Fig. 10.—Alignment chart for solution of equation of a parabola

\[ \phi = P \cdot H \]

\[ D = \text{Diameter of point} \ H \text{feet from top of tree.} \]

\[ H = \text{Distance from top of tree to point where} \ D \text{is taken.} \]

\[ P = \text{Parameter. Constant in trees of same size and form.} \]

\[ R = \text{Exponent. Constant in trees of same form.} \]

\[ FQ = \text{Form quotient} \ D \text{of point halfway between} \ B. \ H. \text{and top} + D \ B \ H. \text{Constant in trees of same form.} \]
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