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Engineering Analysis of Mechanical Damage to Fruits and Vegetables Resulting from Impact

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Engineering Analysis of Mechanical Damage to Fruits and Vegetables Resulting from Impact.

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הספדיה המרכזית
למחקר חקלאי
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- 1) Fruit - Harvesting
- 2) Fruit - Handling
- 2) Fruit grading machinery.

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ABSTRACT

This research was aimed to study the bruising mechanism of apples - as a representative commodity - subjected to dynamic loading, and the associated failure criterion. It consisted of experimental and theoretical aspects, where apples were characterized both as elastic and viscoelastic materials and subjected to quasi-static and impact loading.

The force-deformation response of the quasi-static loading was characterized by a multiple bio-yield points where the occurrence of the first bioyielding with the first bruising, manifested in a form of a brown layer and a determinable bruise volume.

The Hertz model was well fitted as a first approximation to apple response under quasi-static and dynamic loading.

A finite element computer program, with the consideration of inertial effect, has been developed to simulate impact loading by a dynamic analysis instead of static analysis. A comparison between the numerical and analytic solutions shows an excellent agreement.

The mechanism governing bruise was found to be the bruise volume Vs. peak resisting force-input and absorbed energies-impulse relationship.

1. Research Report

1.1 Introduction

One of the major causes of mechanical damage incurred in agricultural commodities is attributed to frequent impacts they receive in harvesting and handling. Fruits and vegetables in particular experience numerous collisions from the time they are harvested until they are processed or consumed.

The frequency and severity of impacts that fruits and vegetables experience increased in recent years with the advance of mechanical harvesters and bulk handling systems. It became obvious, that basic knowledge of the response of these commodities to impact forces is vital in order to minimize mechanical damage incurred in harvesting and handling operations.

The objectives of the study were: to investigate the mechanism which governs impact in soft fruits, represented by apples, and to investigate the failure phenomenon of whole fruit subjected to impact loading and to propose the associated failure criterion.

1.2 Scope of the research

The apples were considered both as an elastic and viscoelastic materials. The assumption of elasticity facilitates the analysis and is accepted as a first approximation since the inaccuracy is tolerable because the applied load occurs over a short time period.

The fruit was subjected to both quasi-static and dynamic loading. Quasi-static loading following Hertz contact theory is valid for depicting impact loading when the duration of impact is very long in comparison with the period of lowest mode of vibration of the two bodies in contact.

Experimentations have shown that the duration of impact is such that the effect of wave propagation may be ignored without loss of accuracy. Nevertheless,

since experimented results of stress wave effect varied as apple variety and test type, higher strain rate loading (up to 170 cm/sec which is equivalent to the end velocity of 15 cm free fall) were also performed and compared with previous results. Moreover, viscoelastic properties measured under quasi-static loading are not appropriate for the study of response of fruits to high rate such as impact.

In addition to the experimentd approach, aimed to study the bruising mechanism, model was developed with consideration of inertial effect to solve the dynamic problem.

Finally, scanning electron microscope (SEM) was employed to study the microstructure of the tissue under loading for a better understanding of the failure mechanism.

1.3 Quasi-static loading

Quasi-static compression was applied through the loading of a rigid flat surface upon apple fruits. Three parameters were considered : physical conditions (newly harvested or stored apples; loading rates (50-100mm/min) and varieties ("Golden Delicious" and "Granny Smith").

The analysis was based on Hertz contact model and was implemented by statistics and computer to solve the behavior of the apple under such loading.

The experimental results and statistical analysis augmented by SEM observations showed several interesting features that should add to our understanding of the damage mechanism.

The force-deformation response was characterized by multiple bioyield points corresponding to a layered bruising pattern. A linear relationship was found between the cell-bursting episodes as indicated by the discontinuities on the force-deformation curve(bioyield) and the number of bruise layers. The first bioyield point caused the first bruising in the form of a brown layer and a calculated bruise-volume. Further loading gave rise to an equal number of bioyield points and bruises layers.

The first bioyield point signals the failure threshold since no bruise was observed up to this point. Loading and unloading within the range increasing to the failure threshold caused generally uninspectable absorbed energy and bruise within this range; therefore, the fruit behaves elastically and lends itself to analysis based on Hertz contact problem. Details pertaining to the experimental procedure and analysed results are given by Jarimopas (1984) and Jarimopas et al (1984).

1.4 Dynamic loading

Two methods were employed to study the response of apples to dynamic loading. An impact loading system was developed consisting of a pendulum type impact generating unit and a dedicated instrumentation system.

The instrument systems comprised of a network of circuits hooking up electronic equipment to detect, measure, display and record mechanical parameters (such as impact force and deformation) relating to the response of apple under impact loading (details of the experimental set-up are given by Jarimopas (1984)). The experiments were performed on "Golden Delicious" apples applying six levels of input energy (approximately 10, 50, 100, 150, 300 and 500 percent) of the pendulum kinetic energy anticipated to cause the first bioyielding at the apple cheek. Variation of the kinetic energy was practically achieved by changing the angle of pendulum bruising and bruise volume were measured and calculated to be compared with that of the quasi-static test. The SEM technique was also applied to compare tissues of apple flesh that were damaged by impact loading with that by quasi-static loading.

The pattern of the force-deformation response appeared similar to that found in the quasi-static test with a discontinuity which may be regarded as a bioyield point. When the kinetic energy of the pendulum was increased by increasing the angle, the apple responded with more bioyield points similar to the response of the quasi-static test. In contrast, if the pendulum was dropped from a small angle, no bioyielding was found.

The Hertz contact model was well fitted to the force-deformation data and hence, the apple could be inferred to be elastic till the bioyield point.

The damage of apple flesh appeared surprisingly as bruise layers whose numbers conform to the number of bioyield points. This result was supported by cell structure investigation through SEM, showing bruise layers having configuration similar to that observed in the quasi-static loading response.

As mentioned before, viscoelastic properties measured under quasi-static loading are not appropriate for the study of response of fruits to high strain rate such as impact. Therefore, an apparatus which is capable of measuring the stress-relaxation response of fruits at high strain rate loadings has been developed. An air cylinder was used to achieve fast loading speeds, and a computer system with high speed data acquisition capability was used to record experimental data. A velocity sensor which consists of LED (light-emit-diode) and photodiode was used to measure the impact velocities and served as a trigger for recording systems.

Experiments with 'golden delicious' apples have been conducted at several impact velocities (up to 170 cm/sec which is equivalent to the end velocity of 15 cm free fall). The experimental data fit the general Maxwell model. The result also shows that the strength of the fruit tissue depends mainly on the stress in the most slowly relaxing element among Maxwell elements. These experiments also provide the necessary information for the analysis of stress distribution of fruits under impact loadings. (Details of the experiments procedure, the apparatus construction and experimental results will be reported in a paper to be presented in the ASAE Winter Meeting 1984).

A mathematical model was developed to include inertial effect which is - at least theoretically - an important factor to be considered. The model was so developed that impact loading was simulated by a dynamic analysis instead of static analysis. A finite element program for studying static stress-strain of axisymmetric viscoelastic bodies has been modified to include the inertial effect. These modifications, an implicit method, Wilson θ method, which is unconditional stable if θ is greater than 1.37 is used to solve this dynamic problem. A comparison between the numerical and analytic solutions shows an excellent agreement. The detailed discussion of model and comparisons are given in Appendix I.

1.5 Analysis of failure

Two approaches of failure criterion were proposed when the apple was treated as an elastic material. Approach I, is applicable whenever loading exceeded infinitesimally the first bioyielding or failure point. The apple instantly failed, and the failure criterion could be defined either in terms of input or the absorbed energy, averaged at the moment of failure. Approach II, resulting from Hertz model fitting, suggests as failure criterion the maximum normal strain at a given apple depth and corresponding to a range of apple diameters.

Analysis of failure criteria of fruit tissue derived from viscoelastic response at impact are still being investigated and will be based on the comparison between computer predicted results and experimental observations of damage. The detailed discussions of these aspects are under preparations and will be presented in two papers to be presented in the ASAE Winter Meeting, 1984.

2. Description of Cooperation

The cooperation between the Israeli and American team was planned to be of a complementary nature rather than following the same procedure under different conditions. Thus, the Israeli team considered the apple as an elastic material, which is valid as a first approximation, while the American team studied the viscoelastic response of the apple, where the effect of strain rate is considered. The work in Israel considered the quasi-static Loading, based on the common agreement that the effect of the wave propagation is small when the duration of impact is large compared to the period of the lowest natural frequency of either body in contact. The American team, on the other hand, performed high strain rate loadings, since viscoelastic properties measured under quasi-static loading are not appropriate for the study of response of fruits to impact.

Relevant experimental techniques were developed respectively to enable the derivation of the desired response under the prescribed conditions.

While the Israeli part was primarily an experimental, following known and accepted models for describing the fruit response, the American team developed a mathematical model with consideration of inertial effect.

Finally, an analysis was made by both teams, of failure criteria of fruit tissue based on the experimental results, and compared with the computer predicted results from the theoretical analysis.

3. Benefit to Agriculture

The increased use of mechanization in harvesting and handling of fruits and vegetables has led to increased mechanical damage in these commodities. This damage is attributed primarily to the frequent impact encountered in the various stages from productions to consumption.

Much research has been done on the problem of impact on fruits and vegetables analyzing both the contact phenomenon and stress wave propagation, which are the major parameters associated with impact. However, despite the impressive amount of research work encompassing both experimental and theoretical approaches, the impact phenomenon is not yet fully understood. The results from various sources are inconclusive and non-comparable, and so far have not provided the basis for defining the causes of damage and establishing a failure criterion.

The results of research constitute an attempt to provide answers to these questions. It is believed that a better understanding was obtained of the mechanics of tissue failure in apples subjected to impact. This understanding will contribute, it is hoped, to the development of better methods of harvesting and handling in terms of a lower rate of mechanical damage. In addition it may lead to improvements in the mechanical strength of the commodity through breeding and chemical treatments. Thus, the contribution toward a decrease in the mechanical damage would benefit the grower, the processor, the re trailer and finally the consumer.

4. List of Publications

- 1 - Chen, P. and Z. Sun (1981) Impact parameters related to bruise injury in apples. ASAE paper no. 81-3041, presented at the ASAE Summer Meeting, Florida.
- 2 - Chen, P. and Z. Sun (1982) critical strain failure criterion: pros and cons, ASAE paper no. 82-3054 presented at the ASAE Meeting Chicago, Ill.
- 3 - Jarimopas, B. Manor, G. and Sarig, Y. (1984) Mechanical behavior and bruising of apple under quasi-static loading. ASAE paper no. 84-1096, presented at the ASAE Summer Meeting, Knoxville, Te.
- 4 - Jarimopas, B. (1984) Failure of apples under dynamic loading. Unpublished D.Sc. Thesis, Faculty of Agricultural Engineering, Technion, Israel. Institute of Technology, Haifa.

Appendix I

FINITE ELEMENT APPROACH WITH INERTIAL EFFECT FOR LINEAR ISOTROPIC VISCOELASTIC MATERIALS

(A) Stress-Strain Law

The total stress τ_{ij} and strain ϵ_{ij} tensors can be separated into deviatoric, s_{ij} and e_{ij} , and volumetric components, σ and θ as follows:

$$s_{ij} = \tau_{ij} - \sigma \delta_{ij} \quad [1]$$

$$e_{ij} = \epsilon_{ij} - \frac{\theta}{3} \delta_{ij} \quad [2]$$

$$\sigma = \frac{1}{3} \sum_{i=1}^3 \tau_{ii} \quad [3]$$

$$\theta = \sum_{i=1}^3 \epsilon_{ii} \quad [4]$$

$$\text{where } \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad [5]$$

For linear isotropic viscoelastic materials, s_{ij} and σ can be expressed as following expressions (1):

$$s_{ij} = 2 \int_0^t \phi_1 (t-\tau) \frac{\partial e_{ij}}{\partial \tau} d\tau \quad [6]$$

$$\sigma = \int_0^t \phi_2 (t-\tau) \frac{\partial \theta}{\partial \tau} d\tau \quad [7]$$

where ϕ_1 and ϕ_2 are relaxation functions in shear and bulk respectively.

As a comparison, for linear isotropic elastic materials,

$$s_{ij} = 2\mu e_{ij} \quad [8]$$

$$\sigma = k\theta \quad [9]$$

where μ and k are elastic shear and bulk moduli respectively.

The model is assumed that the material is viscoelastic in shear and elastic in bulk (2, 3), i.e.

$$s_{ij} = 2 \int_0^t \phi_1(t-\tau) \frac{\partial e_{ij}}{\partial \tau} d\tau \quad [10]$$

$$\sigma = k\theta \quad [11]$$

A numerical procedure with finite element analysis in space and finite difference analysis in time is used. Refer to Figure 1, assuming the solutions for all past time steps have been found, and $t=t_n$ is being investigated.

At $t = t_n$, eqs. [10] and [11] become,

$$s_{ij}(t_n) = s_{ij_n} = 2 \int_0^{t_n} \phi_1(t_n-\tau) \frac{\partial e_{ij}}{\partial \tau} d\tau \quad [12]$$

$$\sigma_n = k\theta_n \quad [13]$$

Similarly,

$$s_{ij_{n-1}} = 2 \int_0^{t_{n-1}} \phi_1(t_{n-1}-\tau) \frac{\partial e_{ij}}{\partial \tau} d\tau \quad [14]$$

$$\sigma_{n-1} = k\theta_{n-1} \quad [15]$$

Note that eq. [12] can also be written as,

$$s_{ij_n} = 2 \int_0^{t_{n-1}} \phi_1(t_n - \tau) \frac{\partial e_{ij}}{\partial \tau} d\tau + 2 \int_{t_{n-1}}^{t_n} \phi_1(t_n - \tau) \frac{\partial e_{ij}}{\partial \tau} d\tau \quad [16]$$

To perform incremental analysis, let's define

$$\Delta s_{ij_n} = s_{ij_n} - s_{ij_{n-1}} \quad [17]$$

$$\Delta \sigma_n = \sigma_n - \sigma_{n-1} \dots \text{etc.} \quad [18]$$

Therefore, eq. [16] - eq. [14]

$$\begin{aligned} \Delta s_{ij_n} = & 2 \int_0^{t_{n-1}} [\phi_1(t_n - \tau) - \phi_1(t_{n-1} - \tau)] \frac{\partial e_{ij}}{\partial \tau} d\tau \\ & + 2 \int_{t_{n-1}}^{t_n} \phi_1(t_n - \tau) \frac{\partial e_{ij}}{\partial \tau} d\tau \end{aligned} \quad [19]$$

and eq. [13] - eq. [15]

$$\Delta \sigma_n = k(\theta_n - \theta_{n-1}) = k\Delta \theta_n \quad [20]$$

It is reasonable to approximate

$$\frac{\partial e_{ij}}{\partial \tau} \approx \frac{e_{ij_n} - e_{ij_{n-1}}}{\Delta t_n} = \frac{\Delta e_{ij_n}}{\Delta t_n} \quad [21]$$

for the time interval $t_{n-1} < \tau < t_n$

Thus,

$$\Delta s_{ij_n} = 2 \int_0^{t_{n-1}} [\phi_1(t_n - \tau) - \phi_1(t_{n-1} - \tau)] \frac{\partial e_{ij}}{\partial \tau} d\tau$$

$$+ 2 \frac{\Delta e_{ij_n}}{\Delta t_n} \int_{t_{n-1}}^{t_n} \phi_1(t_n - \tau) d\tau \quad [22]$$

define

$$\mu_n = \frac{1}{\Delta t_n} \int_{t_{n-1}}^{t_n} \phi_1(t_n - \tau) d\tau \quad [23]$$

$$s_{ij_{o_n}} = 2 \int_0^{t_{n-1}} [\phi_1(t_n - \tau) - \phi_1(t_{n-1} - \tau)] \frac{\partial e_{ij}}{\partial \tau} d\tau \quad [24]$$

∴ eq. [22] and eq. [20] become

$$\Delta s_{ij_n} = 2 \mu_n \Delta e_{ij_n} + s_{ij_{o_n}} \quad [25]$$

$$\Delta \sigma_n = k \Delta \theta_n \quad [26]$$

ϕ_1 can usually be expressed as

$$\phi_1(t) = \sum_{m=1}^M \alpha_m e^{-\beta_m t} \quad [27]$$

substitute eq. [27] into eq. [23]

$$\mu_n = \frac{1}{\Delta t_n} \int_{t_{n-1}}^{t_n} \left[\sum_{m=1}^M \alpha_m e^{-\beta_m (t_n - \tau)} \right] d\tau \quad [28]$$

$$\boxed{\mu_n = \sum_{m=1}^M \alpha_m J_{mn}} \quad [29]$$

where

$$J_{mn} = \frac{1}{\Delta t_n} \int_{t_{n-1}}^{t_n} e^{-\beta_m(t_n - \tau)} d\tau \quad [30]$$

by changing variable

$$u = -\beta_m(t_n - \tau) \quad [31]$$

$$\text{and } \frac{du}{d\tau} = \beta_m \quad [32]$$

$$\therefore J_{mn} = \frac{1}{\beta_m \Delta t_n} \int_{-\beta_m \Delta t_n}^0 e^u du \quad [33]$$

$$\boxed{J_{mn} = \frac{1}{\beta_m \Delta t_n} [1 - e^{-\beta_m \Delta t_n}]} \quad [34]$$

Substitute eq. [27] into eq. [24]

$$s_{ijo_n} = 2 \sum_{m=1}^M \alpha_m \int_0^{t_{n-1}} [e^{-\beta_m(t_n - \tau)} - e^{-\beta_m(t_{n-1} - \tau)}] \frac{\partial e_{f,j}}{\partial \tau} d\tau \quad [35]$$

$$\begin{aligned} \text{but } e^{-\beta_m(t_n - \tau)} &= e^{-\beta_m t_n} e^{\beta_m t_{n-1}} e^{-\beta_m t_{n-1} - \beta_m \tau} \\ &= e^{-\beta_m \Delta t_n} e^{-\beta_m(t_{n-1} - \tau)} \end{aligned} \quad [36]$$

$$\therefore s_{ij0_n} = 2 \sum_{m=1}^M \alpha_m [e^{-\beta_m \Delta t_n} - 1] C_{ijm_n} \quad [37]$$

where

$$C_{ijm_n} = \int_0^{t_{n-1}} e^{-\beta_m(t_{n-1} - \tau)} \frac{\partial e_{ij}}{\partial \tau} d\tau \quad [38]$$

Note,

$$C_{ijm_{n-1}} = \int_0^{t_{n-2}} e^{-\beta_m(t_{n-2} - \tau)} \frac{\partial e_{ij}}{\partial \tau} d\tau \quad [39]$$

$$\begin{aligned} C_{ijm_n} &= \int_0^{t_{n-2}} e^{-\beta_m(t_{n-1} - \tau)} \frac{\partial e_{ij}}{\partial \tau} d\tau \\ &+ \int_{t_{n-2}}^{t_{n-1}} e^{-\beta_m(t_{n-1} - \tau)} \frac{\partial e_{ij}}{\partial \tau} d\tau \end{aligned} \quad [40]$$

Similar to eq. [21] and eq. [36], there are

$$\frac{\partial e_{ij}}{\partial \tau} \approx \frac{\Delta e_{ij_{n-1}}}{\Delta t_{n-1}}, \quad t_{n-2} < \tau < t_{n-1} \quad [41]$$

$$e^{-\beta_m(t_{n-1} - \tau)} = e^{-\beta_m \Delta t_{n-1}} e^{-\beta_m(t_{n-2} - \tau)} \quad [42]$$

∴ eq. [40] becomes

$$C_{ijm_n} = e^{-\beta_m \Delta t_{n-1}} \int_0^{t_{n-2}} e^{-\beta_m (t_{n-2}-\tau)} \frac{\partial e_{ij}}{\partial \tau} d\tau$$

$$+ \frac{\Delta e_{ij_{n-1}}}{\Delta t_{n-1}} \int_{t_{n-2}}^{t_{n-1}} e^{-\beta_m (t_{n-1}-\tau)} d\tau \quad [43]$$

by using eqs. [39] and [30]

$$C_{ijm_n} = e^{-\beta_m \Delta t_{n-1}} C_{ijm_{n-1}} + \Delta e_{ij_{n-1}} J_{m_{n-1}} \quad [44]$$

(B) Dynamical Analysis

By using stress-strain law and theorem of minimum potential energy with the consideration of inertial forces, the system matrix equation of finite element procedure can be expressed as (4, 5, 6, 7).

$$\underline{\underline{M}} \ddot{\underline{U}} + \underline{\underline{S}} \underline{U} = \underline{R} \quad [45]$$

Where $\underline{\underline{M}}$ and $\underline{\underline{S}}$ are system mass and stiffness matrices, $\ddot{\underline{U}}$, $\underline{U}$ and $\underline{R}$ are system acceleration, displacement and load vectors, note that $\underline{\underline{S}}$ is function of time.

For time interval t to $t + \Delta t$, assume $\underline{\underline{M}}$ and $\underline{\underline{S}}$ remain constant,

$$\underline{\underline{M}}_t \ddot{\underline{U}}_t + \underline{\underline{S}}_t \underline{U}_t = \underline{R}_t \quad [46]$$

$$\underline{\underline{M}}_t \ddot{\underline{U}}_{t+\Delta t} + \underline{\underline{S}}_t \underline{U}_{t+\Delta t} = \underline{R}_{t+\Delta t} \quad [47]$$

eq. [47] - eq. [46]

$$\underline{\underline{M}}_t \Delta \ddot{\underline{U}}_{t+\Delta t} + \underline{\underline{S}}_t \Delta \underline{U}_{t+\Delta t} = \Delta \underline{R}_{t+\Delta t} \quad [48]$$

where

$$\Delta \ddot{\underline{U}}_{t+\Delta t} = \ddot{\underline{U}}_{t+\Delta t} - \ddot{\underline{U}}_t \quad [49]$$

$$\Delta \underline{U}_{t+\Delta t} = \underline{U}_{t+\Delta t} - \underline{U}_t \quad [50]$$

$$\Delta \underline{R}_{t+\Delta t} = \underline{R}_{t+\Delta t} - \underline{R}_t \quad [51]$$

Also the element mass matrix $\underline{\underline{M}}_{ij}^{(e)}$ can be expressed as (4)

$$\underline{\underline{M}}_{ij}^{(e)} = \int_{V(e)} \underline{N}^T \rho \underline{N} dv \quad [52]$$

where $\tilde{N}$ is shape function and ρ is material density.

For axisymmetric body with quadrilateral element and constant density analysis,

$$\tilde{M}^{(e)} = \rho \int_{V(e)} \tilde{N}^T \tilde{N} (r dr dz d\theta) \quad [53]$$

Consider only unit radian (i.e., $d\theta = 1$) and transform coordinates r and z to master element coordinate system ξ and η (see Figure 2), eq. [53] becomes,

$$\tilde{M}^{(e)} = \rho \int_{-1}^1 \int_{-1}^1 \tilde{N}^T \tilde{N} (\sum R_i N_i) |J| d\xi d\eta \quad [54]$$

Where Z_i = nodal z -coordinates

R_i = nodal r -coordinates

$|J|$ = Jacobian

$$= \frac{1}{16} [(C_1 + d\eta)(a_1 + b\xi) - (C_2 + d\xi)(a_2 + b\eta)] \quad [55]$$

$$a_1 = \sum_{i=1}^4 Z_i \eta_i \quad [56]$$

$$a_2 = \sum_{i=1}^4 Z_i \xi_i \quad [57]$$

$$b = \sum_{i=1}^4 Z_i \xi_i \eta_i \quad [58]$$

$$C_1 = \sum_{i=1}^4 R_i \xi_i \quad [59]$$

$$C_2 = \sum_{i=1}^4 R_i \eta_i \quad [60]$$

$$d = \sum_{i=1}^4 R_i \xi_i \eta_i \quad [61]$$

$$N_i = \frac{1}{4} (1 + \xi_i \xi)(1 + \eta_i \eta) \quad [62]$$

As pointed out by Zienkiewicz (4), in application the results of analysis by consistent and lumped mass matrices are almost identical, for simplicity, lumped mass procedure is applied to eq. [54].

Now, Wilson θ method (5, 6, 7, 8) is used to solve eq. [48]. It is basically an extension of linear acceleration method (see Figures 3a and 3b), no special starting procedures are needed, and is an implicit method with unconditional stability if $\theta > 1.37$. In application, θ is usually employed as 1.4 (8).

Refer to Figure 3b, let τ denote the increase in time, where $0 < \tau < \theta\Delta t$, then for the time interval $t < t^* < t + \theta\Delta t$, linear acceleration assumption gives,

$$\ddot{\tilde{U}}_{t^*} = \ddot{\tilde{U}}_t + \frac{t^*-t}{\theta\Delta t} (\ddot{\tilde{U}}_{t+\theta\Delta t} - \ddot{\tilde{U}}_t) \quad [63]$$

define

$$\Delta\ddot{\tilde{U}}_{t+\theta\Delta t} = \ddot{\tilde{U}}_{t+\theta\Delta t} - \ddot{\tilde{U}}_t \quad [64]$$

Eq. [63] becomes,

$$\ddot{\tilde{U}}_{t^*} = \ddot{\tilde{U}}_t + \frac{t^*-t}{\theta\Delta t} (\Delta\ddot{\tilde{U}}_{t+\theta\Delta t}) \quad [65]$$

Integrate eq. [65] from time t to $t+\tau$

$$\int_t^{t+\tau} \ddot{\tilde{U}}_{t^*} dt^* = \int_t^{t+\tau} \left(\ddot{\tilde{U}}_t + \frac{t^*-t}{\theta\Delta t} \Delta\ddot{\tilde{U}}_{t+\theta\Delta t} \right) dt^* \quad [66]$$

$$\left[\begin{array}{c} \dot{\tilde{U}}_{t^*} \\ \tilde{U}_{t^*} \end{array} \right]_t^{t+\tau} = \left[\begin{array}{c} \ddot{\tilde{U}}_t t^* + \frac{(t^*)^2}{2(\theta\Delta t)} \ddot{\Delta\tilde{U}}_{t+\theta\Delta t} - \frac{t t^*}{\theta\Delta t} \ddot{\Delta\tilde{U}}_{t+\theta\Delta t} \\ \tilde{U}_t t^* + \frac{(t^*)^2}{2(\theta\Delta t)} \ddot{\Delta\tilde{U}}_{t+\theta\Delta t} - \frac{t t^*}{\theta\Delta t} \ddot{\Delta\tilde{U}}_{t+\theta\Delta t} \end{array} \right]_t^{t+\tau} \quad [67]$$

$$\dot{\tilde{U}}_{t+\tau} - \dot{\tilde{U}}_t = \ddot{\tilde{U}}_t \tau + \frac{\tau^2}{2(\theta\Delta t)} \ddot{\Delta\tilde{U}}_{t+\theta\Delta t} \quad [68]$$

Define,

$$\Delta\dot{\tilde{U}}_{t+\tau} = \dot{\tilde{U}}_{t+\tau} - \dot{\tilde{U}}_t \quad [69]$$

Eq. [68] becomes,

$$\Delta\dot{\tilde{U}}_{t+\tau} = \ddot{\tilde{U}}_t \tau + \frac{\tau^2}{2(\theta\Delta t)} \ddot{\Delta\tilde{U}}_{t+\theta\Delta t} \quad [70]$$

Similarly, integration of eq. [68] gives

$$\Delta\tilde{U}_{t+\tau} = \dot{\tilde{U}}_t \tau + \frac{1}{2} \ddot{\tilde{U}}_t \tau^2 + \frac{\tau^3}{6\theta\Delta t} \ddot{\Delta\tilde{U}}_{t+\theta\Delta t} \quad [71]$$

At time $t = t + \theta\Delta t$ (i.e., $\tau = \theta\Delta t$)

Eqs. [70], [71] become,

$$\Delta\dot{\tilde{U}}_{t+\theta\Delta t} = \ddot{\tilde{U}}_t (\theta\Delta t) + \frac{\theta\Delta t}{2} \ddot{\Delta\tilde{U}}_{t+\theta\Delta t} \quad [72]$$

$$\Delta\tilde{U}_{t+\theta\Delta t} = \dot{\tilde{U}}_t (\theta\Delta t) + \frac{(\theta\Delta t)^2}{2} \ddot{\tilde{U}}_t + \frac{(\theta\Delta t)^2}{6} \ddot{\Delta\tilde{U}}_{t+\theta\Delta t} \quad [73]$$

Solving for $\dot{\Delta\tilde{U}}_{t+\theta\Delta t}$ and $\ddot{\Delta\tilde{U}}_{t+\theta\Delta t}$ in terms of $\Delta\tilde{U}$ from eqs. [72] and [73] gives,

$$\dot{\Delta \tilde{U}}_{t+\theta\Delta t} = \frac{3}{\theta\Delta t} \Delta \tilde{U}_{t+\theta\Delta t} - 3\dot{\tilde{U}}_t - \frac{\theta\Delta t}{2} \ddot{\tilde{U}}_t \quad [74]$$

$$\ddot{\Delta \tilde{U}}_{t+\theta\Delta t} = \frac{6}{(\theta\Delta t)^2} \Delta \tilde{U}_{t+\theta\Delta t} - \frac{6}{(\theta\Delta t)} \dot{\tilde{U}}_t - 3 \ddot{\tilde{U}}_t \quad [75]$$

By eq. [48], for time interval t to $t+\theta\Delta t$, similarly, we have

$$\ddot{\tilde{M}}_t \Delta \tilde{U}_{t+\theta\Delta t} + \ddot{\tilde{S}}_t \Delta \tilde{U}_{t+\theta\Delta t} = \Delta \tilde{R}_{t+\theta\Delta t} \quad [76]$$

Substituting eqs. [74] and [75] into eq. [76] and rearranging gives

$$\ddot{\tilde{S}}_{t+\theta\Delta t} \Delta \tilde{U}_{t+\theta\Delta t} = \Delta \tilde{R}_{t+\theta\Delta t} \quad [77]$$

Where,

$$\ddot{\tilde{S}}_{t+\theta\Delta t} = \ddot{\tilde{S}}_t + \frac{6}{(\theta\Delta t)^2} \ddot{\tilde{M}}_t \quad [78]$$

$$\Delta \tilde{R}_{t+\theta\Delta t} = \theta \Delta \tilde{R}_{t+\Delta t} + \ddot{\tilde{M}}_t \left[\frac{6}{\theta\Delta t} \dot{\tilde{U}}_t + 3 \ddot{\tilde{U}}_t \right] \quad [79]$$

Note that $\Delta \tilde{R}_{t+\theta\Delta t}$ has been approximated as

$$\Delta \tilde{R}_{t+\theta\Delta t} = \theta \Delta \tilde{R}_{t+\Delta t} \quad [80]$$

From eq. [77], solution at time $t+\theta\Delta t$, $\Delta \tilde{U}_{t+\theta\Delta t}$, can be obtained, it is necessary to interpolate back to time $t+\Delta t$ and have $\Delta \tilde{U}_{t+\Delta t}$ solved.

To do this, substitute $\Delta \tilde{U}_{t+\theta\Delta t}$ into eq. [75] to obtain acceleration increment for $\theta\Delta t$ time interval first.

$$\ddot{\Delta \tilde{U}}_{t+\theta\Delta t} = \frac{6}{(\theta\Delta t)^2} \Delta \tilde{U}_{t+\theta\Delta t} - \frac{6}{\theta\Delta t} \dot{\tilde{U}}_t - 3 \ddot{\tilde{U}}_t \quad [81]$$

The acceleration increment for normal time step Δt can then be obtained by linear interpolation because of linear acceleration assumption:

$$\ddot{\Delta U}_{\sim t+\Delta t} = \frac{1}{\theta} \ddot{\Delta U}_{\sim t+\theta\Delta t} \quad [82]$$

Substituting eq. [82] into eqs. [72] and [73] with $\theta = 1$ gives

$$\dot{\Delta U}_{\sim t+\Delta t} = \ddot{U}_t \Delta t + \frac{\Delta t}{2} \ddot{\Delta U}_{\sim t+\Delta t} \quad [83]$$

$$\Delta U_{\sim t+\Delta t} = \dot{U}_t \Delta t + \frac{(\Delta t)^2}{2} \ddot{U}_t + \frac{(\Delta t)^2}{6} \ddot{\Delta U}_{\sim t+\Delta t} \quad [84]$$

Finally,

$$U_{\sim t+\Delta t} = U_{\sim t} + \Delta U_{\sim t+\Delta t} \quad [85]$$

$$\dot{U}_{\sim t+\Delta t} = \dot{U}_{\sim t} + \dot{\Delta U}_{\sim t+\Delta t} \quad [86]$$

$$\ddot{U}_{\sim t+\Delta t} = \ddot{U}_{\sim t} + \ddot{\Delta U}_{\sim t+\Delta t} \quad [87]$$

(C) Consideration of boundary conditions at time $t+\theta\Delta t$

Force boundary conditions at time $t+\theta\Delta t$ is simply expressed as

$$(F_{BC})_{t+\theta\Delta t} = \theta (F_{BC})_{t+\Delta t} \quad [88]$$

Displacement boundary conditions at time $t+\theta\Delta t$ can be derived from eqs. [82] and [84]. Rearrangement of eqs. [82] and [84] gives

$$(\Delta U_{BC})_{t+\theta\Delta t} = \frac{\theta^3(\Delta t)^2}{6} (\ddot{U}_{BC})_{t+\Delta t} + \theta\Delta t (\dot{U}_{BC})_t + \frac{(\theta\Delta t)^2}{2} (\ddot{U}_{BC})_t \quad [89]$$

$$(\ddot{U}_{BC})_{t+\Delta t} = \frac{6}{(\Delta t)^2} (\Delta U_{BC})_{t+\Delta t} - \frac{6}{\Delta t} (\dot{U}_{BC})_t - 3 (\ddot{U}_{BC})_t \quad [90]$$

Substitute eq. [90] into eq. [89] and rearrange

$$(\Delta U_{BC})_{t+\theta\Delta t} = \theta^3 \left[(\Delta U_{BC})_{t+\Delta t} + \frac{\Delta t}{2} (1-\theta^2) (\dot{U}_{BC})_t + \frac{(\Delta t)^2}{2\theta} (1-\theta) (\ddot{U}_{BC})_t \right] \quad [91]$$

For those boundary displacements are zeroes regardless of time,

$$(\Delta U_{BC})_{t+\theta\Delta t} = (\Delta U_{BC})_{t+\Delta t} = 0 \quad [92]$$

(D) Comparison between numerical and analytic solutions

The analytic solution of wave propagation in elastic half-space subjected to uniform surface tractions (see Figure 4) is used to check the dynamic analysis of numerical procedures.

The analytic solution for normal stress $\tau_y(y,t)$ and particle velocity $\dot{U}(y,t)$ are (9),

$$\tau_y(y,t) = -p(t - \frac{y}{C_L}) \quad [93]$$

$$\dot{U}(y,t) = \frac{C_L}{\lambda + 2\mu} p(t - \frac{y}{C_L}) \quad [94]$$

where

$p(t)$ is a spatially uniform surface pressure

y is half space coordinate

$$C_L = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \text{ dilatation velocity} \quad [95]$$

λ = Lamé's elastic constant

μ = elastic shear modules

ρ = material density

choosing $\rho = 36$
 $E = 30$ (elastic modulus)
 $\nu = 0.25$ (Poisson ratio)
 $\lambda = 12$
 $\mu = 12$
 $\Delta t = 0.1 \pi$ (time increment)

$$C_L = \sqrt{\frac{12+2 \times 12}{36}} = 1$$

and let $p(t) = \begin{cases} 0 & t < 0 \text{ or } t > \pi \\ \sin^2 t & 0 < t < \pi \end{cases}$

$p(t)$ is shown in Figure 5.

Wave propagation in a half space body is simulated by a plain strain analysis of a rectangular parallelepiped with a length of 2π which is divided into 20 elements and with unit width and depth when performing the finite element calculations (see Figure 6).

The numerical results show excellent agreements with analytic solutions in both normal stress and particle velocity comparisons. These comparison are shown in Figures 7, 8, 9 and 10.

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Figure 1. Finite difference in time domain, time steps may not be necessarily the same.

Figure 2. Mapping to master element (ξ - η coord. system) with transformation functions $r = \sum_{i=1}^4 R_i N_i$ and $z = \sum_{i=1}^4 Z_i N_i$, where

$$N_i = \frac{1}{4} (1 + \xi_i \xi) (1 + \eta_i \eta).$$

Figure 3a. Linear acceleration method, $\theta = 1$.

Figure 3b. Wilson θ method, $\theta \geq 1.37$ for stability.

Figure 4. Half-space subjected to surface tractions.

Figure 5. $p(t)$ is a spatially uniform surface pressure, $p(t) = \sin^2 t$ when $0 < t < \pi$ and $p(t) = 0$ elsewhere.

Figure 6a. Boundary conditions and dimension of a finite element domain.

Figure 6b. Finite element mesh. Numbers without circle are nodal numbers, with circle are element numbers.

Figure 7. Comparison of normal stress, τ_y , between analytic and finite element solutions at $y = \pi/20$.

Figure 8. Comparison of normal stress, τ_y , between analytic and finite element solutions at $y = (21/20)\pi$.

Figure 9. Comparison of particle velocity, $\dot{U}_y$, between analytic and finite element solutions at $y = \pi/20$.

Figure 10. Comparison of particle velocity, $\dot{U}_y$, between analytic and finite element solutions at $y = (21/20)\pi$.